Repeatable light paths in the shearfree normal cosmological models

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(Dated:)

Conditions for the existence of repeatable light paths (RLPs) in the shearfree normal cosmological models are investigated. It is found that in the conformally nonflat models the only RLPs are radial null geodesics (in the spherical case) and their analogues in the plane- and hyperbolically symmetric cases. In the conformally flat Stephani models, there exist special spherically-, plane- and hyperbolically symmetric subcases, in which all null geodesics are RLPs. They are slightly more general than the Friedmann – Lemaître – Robertson – Walker models of the corresponding symmetries: their curvature index function $k(t)$ and the scale factor $R(t)$ are expressed through a single function of time. In addition to that, there exist special cases of the Stephani solution in which some of the null geodesics are RLPs. All these special cases are identified.

I. THE MOTIVATION

In a recent paper [1] it was found that in a general Szekeres model [2, 3] of the $\beta' \neq 0$ class there are no repeatable light paths (RLPs). This means, given a fixed light source S and a fixed observer O, two light rays emitted from S in such a direction that they hit O, but at different time instants, intersect different sequences of matter world lines on the way. The observer sees then the light source slowly drifting through the sky. Depending on the mass distribution along the ray, and on the positions of O and S with respect to each other, the average rate of change of the direction toward S measured by O would be between 10^{-8} and 10^{-7} arc sec per year. With the current precision of direction determination equal to 10^{-6} arc sec [1], in the most favourable configuration 10 years of monitoring would be required to measure this effect. Since this drift is strictly zero in the Friedmann – Lemaître – Robertson – Walker (FLRW) models, it can be a qualitative observational test of homogeneity of the Universe.

In Ref. [1] it was found that the only subcase of the $\beta' \neq 0$ Szekeres family of models in which *all* null geodesics are RLPs are the Friedmann models. The condition for all null geodesics to be RLPs was the vanishing of shear in the flow of the cosmic medium, which reduces the Szekeres model to the Friedmann limit. In the general Szekeres models, the matter source moves with zero acceleration (i.e. along timelike geodesics) and zero rotation. This gives rise to the question whether the non-RLP phenomenon is caused by shear or by the model being non-Friedmannian.

This question is addressed in the present paper. The condition of existence of RLPs is applied here to the shearfree normal (SFN) cosmological models [4, 5] – another class of generalisations of the FLRW models, in which the matter source is a perfect fluid moving with zero rotation and zero shear, but nonzero acceleration. In the most general conformally nonflat SFN models, the only RLPs are radial null geodesics (in the spherical case) and their analogues in the plane- and hyperbolically symmetric cases. In the conformally flat Stephani models, there exist special spherically-, plane-, hyperbolically- and axially symmetric subcases, in which all null geodesics are RLPs. They are slightly more general than the FLRW models of the corresponding symmetries: their curvature index function $k(t)$ and the scale factor $R(t)$ are expressed through a single function of time. There also exist special cases of the Stephani model, in which some of the null geodesics are RLPs. For example, in the general axially symmetric case the RLPs are those null geodesics that intersect each space $t =$ constant on the axis of symmetry (in full analogy with the Szekeres models [1]). All these special cases are identified.

Thus, the key to the RLP property is not vanishing shear, but high symmetry.

II. THE SHEARFREE NORMAL (SFN) COSMOLOGICAL MODELS

The SFN models are solutions of Einstein's equations with a perfect fluid source that moves with zero rotation and zero shear, but nonzero expansion and acceleration. The full collection of these models was first found by Barnes [4] in 1973, but special cases were known before (see Ref. [5] for a classification of special cases and the account of historical order). This family of models consists of the conformally flat Stephani Universe [6] found in 1967, and 3 subfamilies of Petrov type D solutions, first found in full generality by Barnes [4]. Each of these type D subfamilies has a 3-dimensional symmetry group acting on 2-dimensional orbits; the symmetry is either spherical or plane or hyperbolic. The spherically symmetric subfamily was first presented by Kustaanheimo and Qvist in 1948 [7], but one special case of it was derived by McVittie already in 1933 [8]. The other two

Petrov type D subfamilies first emerged in the paper by Barnes [4]. This author [9] found a coordinate system that covers all 3 subfamilies, but for the present paper it will be more convenient to consider them separately.

In the Petrov type D case, the metric in comoving coordinates is

$$\mathrm{d}s^2 = \left(\frac{FV_{,t}}{V}\right)^2 \mathrm{d}t^2 - \frac{1}{V^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \quad (2.1)$$

where $F(t)$ is an arbitrary function, related to the expansion scalar θ by $\theta = 3/F$. The Einstein equations reduce to the single equation:

$$w_{,uu}/w^2 = f(u), \quad (2.2)$$

where $f(u)$ is another arbitrary function, while the variable u and the function w are related to the coordinates x, y, z, and to the function $V(t, x, y, z)$ differently in each subfamily. We have

$$(u, w) = \begin{cases} (r^2, V) & \text{with spherical symmetry;} \\ (z, V) & \text{with plane symmetry;} \\ (x/y, V/y) & \text{with hyperbolic symmetry.} \end{cases}$$
$$r^2 \overset{\text{def}}{=} x^2 + y^2 + z^2. \quad (2.3)$$

The formulae for matter density and pressure are known for each case, but will not be used in the present paper, so they are not quoted; see Ref. [5].

The Weyl tensor is proportional to $f(u)$, and so with $f(u) \equiv 0$ the models given by (2.2) become conformally flat. Then they become subcases of the Stephani solution given below, but are still more general than FLRW, see Ref. [5].

The conformally flat Stephani solution [5, 6] has the metric given by (2.1), the coordinates are still comoving, but the function $V(t, x, y, z)$ is given explicitly by

$$V = \frac{1}{R}\left\{1 + \frac{1}{4}k(t)\left[(x - x_0(t))^2 \right.\right.$$
$$\left.\left. + (y - y_0(t))^2 + (z - z_0(t))^2\right]\right\}, \quad (2.4)$$

where (R, k, x_0, y_0, z_0) are arbitrary functions of t. This is easily seen to be a generalisation of the whole FLRW class, to which it reduces when (k, x_0, y_0, z_0) are all constant. The constants (x_0, y_0, z_0) can then be set to zero by a coordinate transformation, the constant k is the FLRW curvature index, and $R(t)$ is the FLRW scale factor. In general, the solution (2.4) has no symmetry.

As with the type D models, the formulae for mass density and pressure are known, but will not be used here, see [5, 6]. The mass density depends only on t.

The parametrisation used in (2.4) follows the original source [6] and is designed so that the most popular representation of the FLRW models is easily obtained from it as a spatially homogeneous limit. It suggests that V is either quadratic in (x, y, z) or does not depend on them. In fact, the metric (2.1) is still conformally flat with

$V = A_0 + A_q(x^2 + y^2 + z^2) + A_1 x + A_2 y + A_3 z$, where the A's are arbitrary functions of t. However, even when $A_q = 0$, the quadratic terms can be restored with use of the Haantjes transformation, see Refs. [9, 10] for the definition and examples of transformations.

As already mentioned, all these solutions have zero shear and zero rotation. The quantity that makes them more general than FLRW is acceleration, which is in every case proportional to the spatial gradient of the g_{00} component in the metric (2.1). Thus, the invariant condition for each of the metrics discussed above to reduce to an FLRW limit is

$$\frac{\partial}{\partial x^i}\left(\frac{V_{,t}}{V}\right) = 0, \quad i = 1, 2, 3. \quad (2.5)$$

Note that the metrics (2.1) do not allow a static limit – $V_{,t}$ must be nonzero. Static solutions that are shear-free and rotation-free obviously exist, but they form a separate branch in the family of solutions of Einstein's equations and cannot be recovered from (2.1).

There exists a very large body of literature on the SFN models, their various subcases and generalisations [5], but it is almost exclusively devoted to finding examples of solutions of (2.2) and of its charged generalisation, $w_{,uu} = f(u)w^2 + g(u)w^3$. A general solution is not known.

Part I
The spherically symmetric models

III. THE GEODESIC EQUATIONS

For the spherically symmetric SFN models we first transform the spatial coordinates of (2.1) to the standard spherical coordinates. Then the geodesic equations in the affine parametrisation are:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \left(\frac{F_{,t}}{F} - \frac{V_{,t}}{V} + \frac{V_{,tt}}{V_{,t}}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$$
$$- \frac{1}{F^2 V V_{,t}}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + u\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}s}\right)^2 + u\sin^2\vartheta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\right]$$
$$+ 2\left(\frac{V_{,tu}}{V_{,t}} - \frac{V_{,u}}{V}\right)\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}u}{\mathrm{d}s} = 0, \quad (3.1)$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + 2r F^2 V_{,t}\left(V_{,tu} - \frac{V_{,t} V_{,u}}{V}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$$
$$- 2\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} - r\left[\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}s}\right)^2 + \sin^2\vartheta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\right]$$
$$+ 2r\frac{V_{,u}}{V}\left[-\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + u\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}s}\right)^2 + u\sin^2\vartheta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\right]$$
$$= 0, \quad (3.2)$$

$$\frac{\mathrm{d}^2\vartheta}{\mathrm{d}s^2} - 2\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\vartheta}{\mathrm{d}s} + \left(\frac{1}{u} - 2\frac{V_{,u}}{V}\right)\frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}\vartheta}{\mathrm{d}s}$$

$$- \cos\vartheta \sin\vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = 0, \tag{3.3}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} - 2\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} + \left(\frac{1}{u} - 2\frac{V_{,u}}{V}\right)\frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s}$$

$$+ 2\cot\vartheta \frac{\mathrm{d}\vartheta}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0. \tag{3.4}$$

Equations (3.1) – (3.2) can be simplified when we use the null condition:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + u\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}s}\right)^2 + u\sin^2\vartheta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = (FV_{,t})^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2. \tag{3.5}$$

What remains of them is:

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \left(\frac{F_{,t}}{F} - 2\frac{V_{,t}}{V} + \frac{V_{,tt}}{V_{,t}}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$$

$$+ 2\left(\frac{V_{,tu}}{V_{,t}} - \frac{V_{,u}}{V}\right)\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}u}{\mathrm{d}s} = 0, \tag{3.6}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2} + \frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 2\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}$$

$$+ \frac{1}{r}\left(1 - 4u\frac{V_{,u}}{V}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 0. \tag{3.7}$$

From (3.3) – (3.4) it follows that the null geodesics are plane, i.e. that the coordinates can be adapted to each *single* geodesic so that it has $\vartheta = \pi/2$ all along. However, in the following we will consider a bundle of geodesics emanating from a common source, and such an adaptation of coordinates would not be useful for that.

In this and the next two sections, the same method as in Ref. [1] will be used, so, for readers' convenience, the relevant short excerpts from there are copied here with suitable modifications.

For further calculations it is more convenient to use the coordinate r as a parameter, which will be non-affine. This is allowed, but with some caution. It is easily seen from (3.7) that a curve obeying it, on which $\mathrm{d}r/\mathrm{d}s = 0$ over some open range of s has $\mathrm{d}t/\mathrm{d}s = 0$ in that range, and so is spacelike (the second possibility, $2uV_{,tu} - V_{,t} = 0$, could be compatible with (2.2) only in the prohibited case $V_{,t} = 0$). However, (3.6) – (3.7) do not guarantee that $\mathrm{d}r/\mathrm{d}s \neq 0$ at all points; isolated points along a null geodesic, at which $\mathrm{d}r/\mathrm{d}s = 0$ can exist. Thus, r can be used as a parameter on null geodesics only on such segments where $\mathrm{d}s/\mathrm{d}r > 0$ or $\mathrm{d}s/\mathrm{d}r < 0$ throughout.

We have, for any coordinate:

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}r^2} + \frac{\mathrm{d}^2r}{\mathrm{d}s^2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}r}. \tag{3.8}$$

Then, from (3.7) we have:

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2\left[-\frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2\right.$$

$$\left. + 2\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}r} - \frac{1}{r}\left(1 - 4u\frac{V_{,u}}{V}\right)\right]. \tag{3.9}$$

Consequently, (3.6) and (3.3) – (3.4) become, using (3.9):

$$\frac{\mathrm{d}^2t}{\mathrm{d}r^2} + \left(4r\frac{V_{,tu}}{V_{,t}} - \frac{1}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}r} + \left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2$$

$$- \frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^3 = 0, \tag{3.10}$$

$$\frac{\mathrm{d}^2\vartheta}{\mathrm{d}r^2} + \frac{\mathrm{d}\vartheta}{\mathrm{d}r}\left[-\frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 + \frac{1}{r}\right]$$

$$- \cos\vartheta\sin\vartheta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2 = 0, \tag{3.11}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \frac{\mathrm{d}\varphi}{\mathrm{d}r}\left[-\frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 + \frac{1}{r}\right]$$

$$+ 2\cot\vartheta\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0. \tag{3.12}$$

IV. THE REDSHIFT EQUATIONS

Consider, in the metric (2.1), two light signals, the second one following the first after a short time-interval τ, both emitted by the same source and arriving at the same observer. The equation of the trajectory of the first signal is

$$(t, \vartheta, \varphi) = (T(r), \Theta(r), \Phi(r)), \tag{4.1}$$

the corresponding equation for the second signal is

$$(t, \vartheta, \varphi) = (T(r) + \tau(r), \Theta(r) + \zeta(r), \Phi(r) + \psi(r)). \tag{4.2}$$

Thus, we have to allow that while the first ray intersects the hypersurface of a given constant value of the r-coordinate at the point $(t, \vartheta, \varphi) = (T, \Theta, \Phi)$, the second ray intersects the same hypersurface at the point $(t, \vartheta, \varphi) = (T + \tau, \Theta + \zeta, \Phi + \psi)$. In general, it arrives at this hypersurface not only later, but also at a different spatial location. Thus, those two rays will not intersect the same succession of intermediate matter worldlines on the way. Note that, since the coordinates used here are comoving, both the source of light and the observer keep their spatial coordinates unchanged throughout history. Given this, and given that a pair of rays emitted by the same source and received by the same observer is considered, $(\zeta, \psi) = (0, 0)$ holds at the point of emission and at the point of reception. However, the second ray is in general emitted in a different direction than the first one, and is received from a different direction by the observer. This means that in a general cosmological model the observed objects should drift across the sky. (See a brief quantitative discussion of this effect in the Lemaître – Tolman model in Ref. [1].) The directions of the two rays will be determined by $(\mathrm{d}\Theta/\mathrm{d}r, \mathrm{d}\Phi/\mathrm{d}r)$ and $(\mathrm{d}\Theta/\mathrm{d}r + \xi(r), \mathrm{d}\Phi/\mathrm{d}r + \eta(r))$, respectively, where $\xi = \mathrm{d}\zeta/\mathrm{d}r$, $\eta = \mathrm{d}\psi/\mathrm{d}r$. It will be assumed here that $(\mathrm{d}\tau/\mathrm{d}r, \zeta, \psi, \xi, \eta)$ are small of the same order as τ, so all

terms nonlinear in any of them and terms involving their products will be neglected.

In writing out the equations of propagation of redshift, the symbol Δ will be used. It denotes the difference between the relevant expression taken at $(t + \tau, r, \vartheta + \zeta, \varphi + \psi)$ and at $(t, r, \vartheta, \varphi)$, *linearized in* (τ, ζ, ψ); i.e. the difference between the value of a given quantity along the second ray and along the first ray, taken at a hypersurface of a given value of the parameter r (and, automatically, given u). For example $F(t + \tau) - F(t) = \Delta F + \mathcal{O}(\tau^2)$, $V(t + \tau, u) - V(t, u) = \Delta V + \mathcal{O}(\tau^2)$, $\Delta(\mathrm{d}\vartheta/\mathrm{d}r) = \xi$. This operation is a generalisation of the calculation by which Bondi [11] derived the redshift equation for radial null geodesics in the Lemaître – Tolman model, see an account of that method in Ref. [1].

Applying the Δ operation to $(3.10) - (3.12)$ we obtain the following general equations of redshift propagation in the spherically symmetric SFN models:[1]

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}r^2} + 4r\left(\frac{V_{,ttu}}{V_{,t}} - \frac{V_{,tt}\,V_{,tu}}{V_{,t}^2}\right)\frac{\mathrm{d}t}{\mathrm{d}r}\tau$$
$$+ \left(\frac{F_{,tt}}{F} - \frac{F_{,t}^2}{F^2} + \frac{V_{,ttt}}{V_{,t}} - \frac{V_{,tt}^2}{V_{,t}^2}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2\tau$$
$$- \frac{F}{r}\left[(2F_{,t}\,V_{,t} + FV_{,tt})(2uV_{,tu} - V_{,t})\right.$$
$$\left. + FV_{,t}(2uV_{,ttu} - V_{,tt})\right]\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^3\tau$$
$$+ \left(4r\frac{V_{,tu}}{V_{,t}} - \frac{1}{r}\right)\frac{\mathrm{d}\tau}{\mathrm{d}r} + 2\left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\tau}{\mathrm{d}r}$$
$$- \frac{3F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2\frac{\mathrm{d}\tau}{\mathrm{d}r} = 0, \qquad (4.3)$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} + \left[-\frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 + \frac{1}{r}\right]\xi$$
$$- \frac{F}{r}\left[(2F_{,t}\,V_{,t} + FV_{,tt})(2uV_{,tu} - V_{,t})\right.$$
$$\left. + FV_{,t}(2uV_{,ttu} - V_{,tt})\right]\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\tau$$
$$- \frac{2F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\frac{\mathrm{d}\tau}{\mathrm{d}r}$$
$$- \cos(2\vartheta)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2\zeta - \sin(2\vartheta)\frac{\mathrm{d}\varphi}{\mathrm{d}r}\eta = 0, \qquad (4.4)$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}r} + \left[-\frac{F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 + \frac{1}{r}\right]\eta$$
$$- \frac{F}{r}\left[(2F_{,t}\,V_{,t} + FV_{,tt})(2uV_{,tu} - V_{,t})\right.$$

$$\left. + FV_{,t}(2uV_{,ttu} - V_{,tt})\right]\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2\frac{\mathrm{d}\varphi}{\mathrm{d}r}\tau$$
$$- \frac{2F^2V_{,t}}{r}(2uV_{,tu} - V_{,t})\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\varphi}{\mathrm{d}r}\frac{\mathrm{d}\tau}{\mathrm{d}r} - \frac{2}{\sin^2\vartheta}\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\frac{\mathrm{d}\varphi}{\mathrm{d}r}\zeta$$
$$+ 2\cot\vartheta\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\eta + 2\cot\vartheta\frac{\mathrm{d}\varphi}{\mathrm{d}r}\xi = 0. \qquad (4.5)$$

Along with these, we can use also the result of Δ acting on the null condition (3.5), transformed to the r-parametrisation, it is

$$\left(2FF_{,t}\,V_{,t}^2 + 2F^2V_{,t}\,V_{,tt}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2\tau + 2F^2V_{,t}^2\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\tau}{\mathrm{d}r}$$
$$= 2u\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\xi + u\sin(2\vartheta)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2\zeta + 2u\sin^2\vartheta\frac{\mathrm{d}\varphi}{\mathrm{d}r}\eta. \quad (4.6)$$

If $\mathcal{T}$ is the period of the electromagnetic wave measured in the rest frame of the source or of the observer, the redshift is given by

$$\frac{\mathcal{T}(r_{\mathrm{obs}})}{\mathcal{T}(r_{\mathrm{em}})} = 1 + z(r_{\mathrm{em}}), \qquad (4.7)$$

where the labels "obs" and "em" refer to the events of observation and emission of the ray, respectively [1, 10]. The period in the rest frame of an object is measured in the units of proper time of that object. In the metrics (2.1) the differential of the proper time, $\mathrm{d}s$, is related to the differential of the coordinate time, $\mathrm{d}t$, by $\mathrm{d}s = \sqrt{g_{00}}\mathrm{d}t$, where $g_{00} = (FV_{,t}/V)^2$. Thus, taking the $\tau(r)$ found from $(4.3) - (4.6)$ as corresponding to the period $\mathcal{T}$, we calculate the redshift from

$$1 + z(r_{\mathrm{em}}) = \frac{\left.(\sqrt{g_{00}}\tau)\right|_{t_{\mathrm{obs}}, r_{\mathrm{obs}}}}{\left.(\sqrt{g_{00}}\tau)\right|_{t_{\mathrm{em}}, r_{\mathrm{em}}}}. \qquad (4.8)$$

Note that eq. (4.3) is decoupled from the other two and determines the redshift independently of (ζ, ψ, ξ, η). This means that even for a non-radial ray the redshift changes with r by the same law as for a radial one. (Indeed, (3.6) is the same for radial and non-radial rays, so Δ acting on it gives the same result in both cases.) This is a consequence of spherical symmetry *and zero shear* in the SFN models. Using eqs. $(4.1) - (4.5)$ and $(5.3) - (5.12)$ of Ref. [1] one can verify that in the Szekeres model the equation (5.12) that determines τ does depend on (ζ, ψ, ξ, η) even in the spherically symmetric subcase, where $\mathcal{E}_{,r} = 0$ and shear is nonzero.

Since $\zeta = \psi = 0$ at the observer, these quantities are not in fact observable. However, ξ and η are in general nonzero at the observer, which implies the change of direction toward the source with time.

V. REPEATABLE LIGHT PATHS (RLPS)

We say that the light paths are repeatable when the rays sent between a given source and a given observer at

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[1] A quick way to calculate $(4.3) - (4.6)$ is to take the differential of the corresponding quantity at constant r and replace $(\mathrm{d}t, \mathrm{d}\vartheta, \mathrm{d}\varphi, \mathrm{d}(\mathrm{d}\vartheta/\mathrm{d}r), \mathrm{d}(\mathrm{d}\varphi/\mathrm{d}r))$ by $(\tau, \zeta, \psi, \xi, \eta)$.

different times always proceed through the same sequence of intermediate particles of the cosmic medium. This means, when the rays are registered at an r-hypersurface of coordinate radius r, they arrive there at the same spatial location (only at different time instants), i.e.:

$$\zeta = \psi = \xi = \eta = 0 \qquad (5.1)$$

all along each ray. Substituting (5.1) in (4.4) – (4.6) one obtains the conditions that have to be obeyed in order that RLPs exist. We first define:

$$\chi(t, u) \overset{\text{def}}{=} \frac{dt}{dr}\tau \left[(2F_{,t}\,V_{,t} + FV_{,tt}\,)\,(2uV_{,tu} - V_{,t}\,)\right.$$
$$\left. + FV_{,t}\,(2uV_{,ttu} - V_{,tt}\,)\right] + 2FV_{,t}\,(2uV_{,ttu} - V_{,t}\,)\frac{d\tau}{dr}$$
$$(5.2)$$

and then the conditions are

$$-\frac{F}{r}\chi\frac{dt}{dr}\frac{d\vartheta}{dr} = 0, \qquad (5.3)$$

$$-\frac{F}{r}\chi\frac{dt}{dr}\frac{d\varphi}{dr} = 0, \qquad (5.4)$$

$$2FV_{,t}\left[(F_{,t}\,V_{,t} + FV_{,tt}\,)\frac{dt}{dr}\tau + FV_{,t}\,\frac{d\tau}{dr}\right]\frac{dt}{dr} = 0. \quad (5.5)$$

Discarding in (5.5) the impossible solutions $dt/dr = 0$ (this would be a spacelike curve) and $FV_{,t} = 0$ (prohibited – see (2.1)) we obtain:

$$\frac{d\tau}{dr} = -\left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\frac{dt}{dr}\tau. \qquad (5.6)$$

Equations (5.3) – (5.4) imply that either $d\vartheta/dr = d\varphi/dr = 0$ or $\chi = 0$. The first case defines a radial null geodesic. This means that radial null geodesics are RLPs, as expected in a spherically symmetric model. Then (5.6) just defines the redshift as a function of r.

To check the other possibility we substitute (5.6) in $\chi = 0$. The result is, after discarding the factor $2u\left(F^2/r\right)(dt/dr)\tau$:

$$V_{,tt}\,V_{,tu} - V_{,t}\,V_{,ttu} = 0. \qquad (5.7)$$

A general solution of this equation is

$$V = \alpha(u)S(t) + \beta(u), \qquad (5.8)$$

where α, β and S are arbitrary functions of their respective arguments. This must be compatible with (2.2). We disregard the cases $\alpha S_{,t} = 0$ because they lead to the impossible condition $V_{,t} = 0$ – see the penultimate paragraph of Sec. II. Then, (5.8) is compatible with (2.2) only when $f = 0$, i.e. when the metric is conformally flat. The conformally flat case belongs to the Stephani class (2.4), which will be discussed in part IV of this paper. However, in that class the spherically symmetric case emerges among many others, and it will be more convenient to discuss it here.[2]

When $f = 0$, (5.8) substituted in (2.2) gives $\alpha_{,uu} = \beta_{,uu} = 0$, i.e.

$$V = A_1 S + B_1 + (A_2 S + B_2)\,r^2. \qquad (5.9)$$

This is in the Stephani class (2.4), with

$$\frac{1}{R(t)} = A_1 S + B_1, \qquad k(t) = 4\frac{A_2 S + B_2}{A_1 S + B_1},$$
$$x_0 = y_0 = z_0 = 0. \qquad (5.10)$$

An FLRW limit results from (5.9) when the zero-acceleration condition (2.5) is fulfilled, i.e. when $A_2 B_1 - A_1 B_2 = 0$. If $A_2 = 0$, then in the FLRW limit $A_1 B_2 = 0$. But $A_1 = A_2 = 0$ is the prohibited case $V_{,t} = 0$, while $B_2 = A_2 = 0$ is the $k = 0$ FLRW model. So, for a general FLRW limit of (5.9) we have $A_2 \neq 0$ and

$$B_1 = A_1 B_2/A_2. \qquad (5.11)$$

Then, in the FLRW limit

$$V = \frac{1}{R(t)}\left(1 + \frac{k}{4}r^2\right), \qquad (5.12)$$

where

$$R = \frac{A_2}{A_1(A_2 S + B_2)}, \qquad k = \frac{4A_2}{A_1}. \qquad (5.13)$$

Since S is arbitrary and (A_1, A_2) can have any signs, this shows that the model defined by (5.9) reproduces the whole FLRW class in the zero-acceleration limit.

In the solution given by (5.9) – (5.10) all null geodesics are RLPs, since the metric obeys (5.7) with no conditions imposed on the vectors tangent to null geodesics.

Thus, the result is:

Corollary 1: In a general spherically symmetric SFN model the only repeatable light paths are radial null geodesics. In the subcase defined by (5.9) all null geodesics are RLPs. This subcase contains the whole FLRW class, but is more general than FLRW because it has nonzero acceleration. It has zero Weyl tensor, but is less general than the spherically symmetric limit of the

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[2] It may be verified that (5.6) is indeed a first integral of (4.3) modulo the RLP conditions. Hint: take a derivative of (5.6) along a null geodesic (defined by $\mathcal{D}/dr = (dt/dr)\partial/\partial t + \partial/\partial r$), then use (3.10) and (5.6) to eliminate d^2t/dr^2 and $d\tau/dr$. The result will be equal to (4.3) provided that:

$$\frac{\tau}{V_{,t}^2}\frac{dt}{dr}(V_{,tt}\,V_{,tu} - V_{,t}\,V_{,ttu}\,)\left[1 - \left(FV\frac{dt}{dr}\right)^2\right] = 0.$$

The expression in ( ) vanishes when (5.7) is fulfilled, the expression in [ ] vanishes in virtue of the null condition when the geodesic is radial. Thus, (5.6) is a first integral of (4.3) when any of the RLP conditions is fulfilled.

Stephani solution (2.4); in the latter $k(t)$ and $R(t)$ are two independent functions.

Part II
The plane symmetric models

VI.   THE GEODESIC EQUATIONS

The scheme of the calculation is here the same as in Sections III – V; only the explicit forms of the equations are different. Thus, we will limit the explanation and the presentation of intermediate expressions to a necessary minimum.

Analogously to the spherical case we note that z can be used as a parameter on open intervals of each geodesic and use the $\mathrm{d}^2z/\mathrm{d}s^2$ geodesic equation to carry out the reparametrisation.[3]   The geodesic equations parametrised by z are then:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}z^2} - F^2 V_{,t}\, V_{,tz} \left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^3 + \left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2$$
$$+2\frac{V_{,tz}}{V_{,t}}\frac{\mathrm{d}t}{\mathrm{d}z} = 0, \tag{6.1}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} - F^2 V_{,t}\, V_{,tz}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 \frac{\mathrm{d}x}{\mathrm{d}z} = 0, \tag{6.2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - F^2 V_{,t}\, V_{,tz}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 \frac{\mathrm{d}y}{\mathrm{d}z} = 0. \tag{6.3}$$

Equations (6.2) and (6.3) show that $(y\mathrm{d}x/\mathrm{d}z - x\mathrm{d}y/\mathrm{d}z)$ is a constant of the motion. This means that the projection of each geodesic on the (x,y) plane is a straight line.

In obtaining the redshift equations via the Δ operation defined in Sec. IV we take $(\zeta,\psi) \stackrel{\text{def}}{=} \Delta(x,y)$ and $(\xi,\eta) \stackrel{\text{def}}{=} (\mathrm{d}/\mathrm{d}z)(\zeta,\psi)$. Since we will not use the redshift equations in full, we do not display them and proceed to the conditions for the RLPs.

VII.   REPEATABLE LIGHT PATHS

We substitute $\zeta = \psi = \xi = \eta = 0$ in the redshift equations and in the null condition, and obtain the conditions for RLPs:

$$\chi(t,z) \stackrel{\text{def}}{=} \left[(2F_{,t} V_{,t} + FV_{,tt}) V_{,tz} + FV_{,t} V_{,ttz}\right]\frac{\mathrm{d}t}{\mathrm{d}z}\tau$$

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[3] If $\mathrm{d}z/\mathrm{d}s = 0$ on an open interval of the geodesic, then either (a) $\mathrm{d}t/\mathrm{d}s = 0$, which can hold only on a spacelike geodesic, or (b) $V_{,t} = 0$, which is a prohibited case (see the remark below (2.5)), or (c) $V_{,tz} = 0$, which is compatible with (2.2) only when $V_{,t} = 0$ or $f = 0$. The latter case is the conformally flat model that will be dealt with in Sec. VII.

$$+2FV_{,t}\, V_{,tz}\,\frac{\mathrm{d}\tau}{\mathrm{d}z}, \tag{7.1}$$

$$-F\chi\frac{\mathrm{d}t}{\mathrm{d}z}\frac{\mathrm{d}x}{\mathrm{d}z} = 0, \tag{7.2}$$

$$-F\chi\frac{\mathrm{d}t}{\mathrm{d}z}\frac{\mathrm{d}y}{\mathrm{d}z} = 0, \tag{7.3}$$

$$2FV_{,t}\frac{\mathrm{d}t}{\mathrm{d}z}\left[(F_{,t} V_{,t} + FV_{,tt})\frac{\mathrm{d}t}{\mathrm{d}z}\tau + FV_{,t}\frac{\mathrm{d}\tau}{\mathrm{d}z}\right] = 0. \tag{7.4}$$

Discarding the impossible cases, there are two sets of solutions to (7.2) – (7.3): (a) $\mathrm{d}x/\mathrm{d}z = \mathrm{d}y/\mathrm{d}z = 0$ – the analogues of radial null geodesics; (b) $\chi = 0$ – the subset of the plane symmetric models in which all null geodesics are RLPs. Using (7.4) in $\chi = 0$ to eliminate $\mathrm{d}\tau/\mathrm{d}z$, we obtain:

$$V_{,tt}\, V_{,tz} - V_{,t}\, V_{,ttz} = 0, \tag{7.5}$$

a general solution of which is

$$V = \alpha(z)S(t) + \beta(z), \tag{7.6}$$

where α, β and S are arbitrary functions of their respective arguments. Just as (5.8), this is compatible with (2.2) only when the metric becomes conformally flat. Then $\alpha_{,zz} = \beta_{,zz} = 0$ and

$$V = A_1 S + B_1 + (A_2 S + B_2)\, z. \tag{7.7}$$

This case is not covered by the parametrisation of (2.4), but can be brought into the form of (2.4) by the following special Haantjes transformation:

$$(x,y,z) = \frac{\left(x', y', z' + C\left(x'^2 + y'^2 + z'^2\right)\right)}{\mathcal{T}}, \quad \text{where}$$
$$\mathcal{T} \stackrel{\text{def}}{=} 1 + 2Cz' + C^2\left(x'^2 + y'^2 + z'^2\right), \tag{7.8}$$

where C is an arbitrary constant, a group parameter of the transformation (see Refs. [9, 10] for hints on how to handle the calculations). The result of (7.8) is

$$V = C\left[C\left(A_1 S + B_1\right) + A_2 S + B_2\right]\left(x'^2 + y'^2 + z'^2\right)$$
$$+ \left[2C\left(A_1 S + B_1\right) + A_2 S + B_2\right] z' + A_1 S + B_1. \tag{7.9}$$

For later reference we will need (7.9) cast explicitly in the form (2.4). The functions appearing in (2.4) are in our present case:

$$\frac{1}{R} = -\frac{(A_2 S + B_2)^2}{4\gamma}, \tag{7.10}$$

$$\gamma \stackrel{\text{def}}{=} C^2\left(A_1 S + B_1\right) + C\left(A_2 S + B_2\right), \tag{7.11}$$

$$k = -\frac{16\gamma^2}{(A_2 S + B_2)^2}, \tag{7.12}$$

$$z_0 = \frac{1}{\gamma}\left[C\left(A_1 S + B_1\right) + \frac{1}{2}\left(A_2 S + B_2\right)\right]. \tag{7.13}$$

Equation (7.12) suggests that $k < 0$ necessarily. However, the $k = 0$ case follows directly from (7.7), it is

$A_2 = B_2 = 0$, and then $1/R = A_1 S + B_1$. Thus, the plane symmetric subcase of the Stephani model can reproduce the $k \leq 0$ FLRW limits.

The general prescription for the FLRW limit can be found by substituting (7.7) in the zero-acceleration condition (2.4); the result is

$$A_1 B_2 - A_2 B_1 = 0. \qquad (7.14)$$

We will need the formulae for R and k parametrised by z_0 rather than by S. From (7.13) we find

$$S = \frac{B_1 C + B_2/2 - (B_1 C + B_2) C z_0}{(C A_1 + A_2) C z_0 - C A_1 - A_2/2}, \qquad (7.15)$$

and then from (7.12) and (7.10)

$$\frac{1}{2K} = \frac{(C A_1 + A_2) C z_0 - C A_1 - A_2/2}{C^2 (A_2 B_1 - A_1 B_2)}, \quad (7.16)$$

$$K \stackrel{\text{def}}{=} \frac{k}{4R}, \qquad (7.17)$$

$$\frac{1}{2KR} = -\frac{1}{2} z_0{}^2 + \frac{z_0}{C} - \frac{1}{2C^2}. \qquad (7.18)$$

Equations (7.16) and (7.18) will be needed to recognise the plane symmetric subcase among the multitude of cases discussed in Appendix A.

Note that the form (7.16) – (7.18) does not allow taking the FLRW limit (7.14). This is logical, since in this limit z_0 becomes constant while K and R do not, so z_0 cannot be used as a parameter.

We thus have:

Corollary 2: In a general plane symmetric SFN model represented as in (2.1) – (2.3) the only repeatable light paths are null geodesics on which x and y are constant. In the subcase defined by (7.7), which includes the $k \leq 0$ FLRW subclass, all null geodesics are RLPs. This subcase is conformally flat, but less general than the conformally flat limit of the plane symmetric case.

The condition $k \leq 0$ is consistent with what is known about the relation between plane symmetric models and their FLRW limits [5].

Part III
The hyperbolically symmetric models

VIII.   THE GEODESIC EQUATIONS

There is a certain complication in discussing this case: either the metric can be similar to (2.1) or the equation similar to (2.2), but not both things at once. We choose to make the metric similar. We use eq. (2.1) of Ref. [9], rename the functions by $(3\theta, Y, b) \stackrel{\text{def}}{=} (1/F, V, f)$ and transform the r-coordinate used there by $r = \exp(r')$.

Dropping the prime, the metric then becomes:

$$\mathrm{d}s^2 = \left(\frac{FV_{,t}}{V}\right)^2 \mathrm{d}t^2 - \frac{1}{V^2} \left(\mathrm{d}r^2 + \mathrm{d}\vartheta^2 + \sinh^2 \vartheta \mathrm{d}\varphi^2\right), \tag{8.1}$$

and the function $V(t,r)$ must obey:

$$V_{,rr} = f(r)V^2 - V. \qquad (8.2)$$

See [9] for the transformation from (8.1) to (2.1) – (2.3).

Just as in Sec. VI we will now skip most of the explanation and of the intermediate expressions because the calculations exactly parallel those for the spherically symmetric models presented in Sections III – V.

Using the $\mathrm{d}^2 r/\mathrm{d}s^2$ geodesic equation we change the parameter to the non-affine r and obtain:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}r^2} + 2\frac{V_{,tr}}{V_{,t}} \frac{\mathrm{d}t}{\mathrm{d}r} + \left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2$$
$$- F^2 V_{,t} V_{,tr} \left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^3 = 0, \qquad (8.3)$$

$$\frac{\mathrm{d}^2 \vartheta}{\mathrm{d}r^2} - F^2 V_{,t} V_{,tr} \frac{\mathrm{d}\vartheta}{\mathrm{d}r}\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 - \cosh \vartheta \sinh \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2$$
$$= 0, \qquad (8.4)$$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}r^2} - F^2 V_{,t} V_{,tr} \frac{\mathrm{d}\varphi}{\mathrm{d}r}\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 + 2\coth \vartheta \frac{\mathrm{d}\vartheta}{\mathrm{d}r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0. \qquad (8.5)$$

We skip through the general redshift equations and proceed to the RLP equations, where this time $(\zeta, \psi) \stackrel{\text{def}}{=} \Delta(\vartheta, \varphi)$ and $(\xi, \eta) \stackrel{\text{def}}{=} (\mathrm{d}/\mathrm{d}z)(\zeta, \psi)$.

IX.   REPEATABLE LIGHT PATHS

When $\zeta = \psi = \xi = \eta = 0$, the redshift equations become:

$$\chi(t,r) \stackrel{\text{def}}{=} [(2F_{,t} V_{,t} + FV_{,tt})V_{,tr} + FV_{,t} V_{,ttr}]\frac{\mathrm{d}t}{\mathrm{d}r}\tau$$
$$+ 2FV_{,t} V_{,tr} \frac{\mathrm{d}\tau}{\mathrm{d}r}, \qquad (9.1)$$

$$-F\chi \frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\vartheta}{\mathrm{d}r} = 0, \qquad (9.2)$$

$$-F\chi \frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0, \qquad (9.3)$$

$$2FV_{,t}\left[(F_{,t} V_{,t} + FV_{,tt})\frac{\mathrm{d}t}{\mathrm{d}r}\tau + FV_{,t}\frac{\mathrm{d}\tau}{\mathrm{d}r}\right]\frac{\mathrm{d}t}{\mathrm{d}r} = 0. \qquad (9.4)$$

As before, (9.4) implies:

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\frac{\mathrm{d}t}{\mathrm{d}r}\tau. \qquad (9.5)$$

Equations (9.2) – (9.3) tell us that either $\mathrm{d}\vartheta/\mathrm{d}r = \mathrm{d}\varphi/\mathrm{d}r = 0$ or $\chi = 0$. The first case defines a pseudo-radial null geodesic. The other possibility is $\chi = 0$. Using (9.5) we get

$$V_{,tt}\, V_{,tr} - V_{,t}\, V_{,ttr} = 0. \qquad (9.6)$$

A general solution of this equation is

$$V = \alpha(r)S(t) + \beta(r), \qquad (9.7)$$

where α, β and S are arbitrary functions of their respective arguments. As before, this is compatible with (8.2) only when $f = 0$ and the solution becomes conformally flat. Then, from (8.2), V must obey, $V_{,rr} + V = 0$, and the solution is

$$V = (A_1 S + B_1)\sin r + (A_2 S + B_2)\cos r. \qquad (9.8)$$

This is transformed to the standard parametrisation (2.4) by the following chain of transformations.

We first transform the 2-dimensional metric $(\mathrm{d}\vartheta^2 + \sinh^2\vartheta\,\mathrm{d}\varphi^2)$ into $(\mathrm{d}\tau^2 + e^{2\tau}\mathrm{d}z^2)$. An explicit prescription for this is given in Appendix A to Ref. [9]. This does not affect V because ϑ and φ are not present in it. Then we carry out the transformation

$$r = \arcsin\frac{y}{\sqrt{x^2+y^2}}, \qquad \tau = -\frac{1}{2}\ln\left(x^2+y^2\right). \quad (9.9)$$

The metric then becomes:

$$\mathrm{d}s^2 = \left(\frac{FW_{,t}}{W}\right)^2\mathrm{d}t^2 - \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{W^2}, \qquad (9.10)$$

where

$$W \stackrel{\text{def}}{=} \sqrt{x^2+y^2}\,V = (A_1 S + B_1)\,y + (A_2 S + B_2)\,x. \qquad (9.11)$$

The metric (9.10) – (9.11) is within the Stephani class, as can be seen by carrying out the following Haantjes transformation:

$$(x,y,z) = \frac{\left(x' + C\left(x'^2 + y'^2 + z'^2\right), y', z'\right)}{1 + 2Cx' + C^2\left(x'^2 + y'^2 + z'^2\right)}. \qquad (9.12)$$

After this, the metric acquires the form (2.1) with V replaced by

$$\begin{aligned} W_1 &= C\left(A_2 S + B_2\right)\left(x'^2 + y'^2 + z'^2\right) \\ &\quad + (A_2 S + B_2)\,x' + (A_1 S + B_1)\,y'. \end{aligned} \qquad (9.13)$$

When W_1 is cast in the form of (2.4), we obtain

$$x_0 = -\frac{1}{2}, \qquad (9.14)$$

$$y_0 = -\frac{1}{2}\frac{A_1 S + B_1}{A_2 S + B_2}, \qquad (9.15)$$

$$K \stackrel{\text{def}}{=} \frac{k}{4R} = C\left(A_2 S + B_2\right), \qquad (9.16)$$

$$\frac{1}{R} = -\frac{C}{4}\times\frac{(A_1 S + B_1)^2 + (A_2 S + B_2)^2}{A_2 S + B_2}. \qquad (9.17)$$

The constant x_0 can be transformed to $x_0 = 0$.

The FLRW limit follows when $A_1 B_2 - A_2 B_1 = 0$. From (9.16) – (9.17) we find

$$k = -\frac{16\left(A_2 S + B_2\right)^2}{(A_1 S + B_1)^2 + (A_2 S + B_2)^2}, \qquad (9.18)$$

which shows that $k < 0$ necessarily. The V given by (9.8) cannot be made independent of r, so the case $k = 0$ is not contained in this model.

For later reference we will need the formulae for K and R parametrised by y_0 rather than by S. We have

$$S = -\frac{B_1 + 2B_2 y_0}{A_1 + 2A_2 y_0}, \qquad (9.19)$$

$$\frac{1}{2K} = \frac{A_1 + 2A_2 y_0}{2C\left(A_1 B_2 - A_2 B_1\right)}, \qquad (9.20)$$

$$\frac{1}{2KR} = -\frac{1}{2C}y_0^2 - \frac{1}{8C}. \qquad (9.21)$$

Similarly to the plane symmetric model, in this form the FLRW limit $A_1 B_2 - A_2 B_1 = 0$ cannot be taken, for the same reason: in the FLRW models y_0 is constant, while K and R are not, so y_0 cannot be used as a parameter.

Thus we have:

Corollary 3: In a general hyperbolically symmetric SFN model the only repeatable light paths are the analogues of radial null geodesics, on which ϑ and φ in (8.1) are constant. In the subcase defined by (9.10) – (9.11), which includes the FLRW models with $k < 0$, all null geodesics are RLPs. This subcase is conformally flat, but less general than the conformally flat limit of (8.1).

Part IV
The Stephani model

X. THE GEODESIC EQUATIONS

The metric of this model is (2.1), with V given by (2.4). We introduce the abbreviation:

$$D \stackrel{\text{def}}{=} F(t)V_{,t}/V. \qquad (10.1)$$

The geodesic equations in the affine parametrisation, with the null condition already incorporated, are:

$$\begin{aligned} \frac{\mathrm{d}^2 t}{\mathrm{d}s^2} &+ 2\left(\frac{D_{,x}}{D}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{D_{,y}}{D}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{D_{,z}}{D}\frac{\mathrm{d}z}{\mathrm{d}s}\right)\frac{\mathrm{d}t}{\mathrm{d}s} \\ &+ \left(\frac{D_{,t}}{D} - \frac{V_{,t}}{V}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 0, \end{aligned} \qquad (10.2)$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + F^2 V_{,t}\, V_{,tx}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 2\frac{V_{,x}}{V}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2$$

$$- 2\left(\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{V_{,y}}{V}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{V_{,z}}{V}\frac{\mathrm{d}z}{\mathrm{d}s}\right)\frac{\mathrm{d}x}{\mathrm{d}s} = 0, \quad (10.3)$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + F^2 V_{,t} V_{,ty}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 2\frac{V_{,y}}{V}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2$$
$$- 2\left(\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{V_{,x}}{V}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{V_{,z}}{V}\frac{\mathrm{d}z}{\mathrm{d}s}\right)\frac{\mathrm{d}y}{\mathrm{d}s} = 0, \quad (10.4)$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + F^2 V_{,t} V_{,tz}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 2\frac{V_{,z}}{V}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2$$
$$- 2\left(\frac{V_{,t}}{V}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{V_{,x}}{V}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{V_{,y}}{V}\frac{\mathrm{d}y}{\mathrm{d}s}\right)\frac{\mathrm{d}z}{\mathrm{d}s} = 0. \quad (10.5)$$

Note that if any of $\mathrm{d}x^i/\mathrm{d}s$, $i = 1, 2, 3$ (where (x^1, x^2, x^3) $\stackrel{\text{def}}{=} (x, y, z)$) is zero along an open interval of a null geodesic, then (10.2) – (10.5) imply either $\mathrm{d}t/\mathrm{d}s = 0$ in the same interval (which is impossible on a null curve) or $V_{,ti} = 0$ for the respective i. This second possibility must be investigated. Suppose first that

$$V_{,tx} = V_{,ty} = V_{,tz} = 0. \quad (10.6)$$

This means:

$$\left(\frac{k}{2R}\right)_{,t} = x_{0,t} = y_{0,t} = z_{0,t} = 0. \quad (10.7)$$

The metric is then spherically symmetric and the constants (x_0, y_0, z_0) can be set equal to zero by coordinate transformations. This case was dealt with at the end of Sec. V, so we disregard it here.

Hence, we may assume that at least one of the quantities in (10.6) is nonzero. Since a general Stephani metric does not change its form under any permutation of the (x, y, z) coordinates (accompanied by a suitable renaming of the (x_0, y_0, z_0) functions), we may assume without loss of generality that $V_{,tz} \neq 0$, and then (10.5) shows that $\mathrm{d}z/\mathrm{d}s$ cannot be zero over an open interval of a null geodesic. Consequently, z can be chosen as a (non-affine) parameter, with an analogous cautionary remark to the one given above (3.8).

Consequently, using (10.5) and (3.8) for $\mathrm{d}^2 z/\mathrm{d}s^2$ we change the parameter to z in (10.2) – (10.4) and obtain:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}z^2} - F^2 V_{,t} V_{,tz}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^3 + \left(\frac{F_{,t}}{F} + \frac{V_{,tt}}{V_{,t}}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2$$
$$+ 2\left(\frac{V_{,tx}}{V_{,t}}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{V_{,ty}}{V_{,t}}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{V_{,tz}}{V_{,t}}\right)\frac{\mathrm{d}t}{\mathrm{d}z} = 0, \quad (10.8)$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} - F^2 V_{,t} V_{,tz}\frac{\mathrm{d}x}{\mathrm{d}z}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 + F^2 V_{,t} V_{,tx}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 = 0, \quad (10.9)$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - F^2 V_{,t} V_{,tz}\frac{\mathrm{d}y}{\mathrm{d}z}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 + F^2 V_{,t} V_{,ty}\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 = 0, \quad (10.10)$$

XI. REPEATABLE LIGHT PATHS

In this case we skip the redshift equations because they are complicated and voluminous, while we are not going to make any direct use of them. In this section, the meaning of the symbols in the Δ operation is

$$\Delta(t, x, y, \mathrm{d}x/\mathrm{d}z, \mathrm{d}y/\mathrm{d}z) \stackrel{\text{def}}{=} (\tau, \zeta, \psi, \xi, \eta). \quad (11.1)$$

We proceed to the RLP conditions that are obtained by applying the Δ operation to (10.9) – (10.10) and immediately assuming $\zeta = \psi = \xi = \eta = 0$.[4] We do not consider the result of Δ acting on (10.8) because it is the equation for τ that will define the redshift propagation along the emergent RLP, and it does not lead to any limitation on the metric.

The null condition in the z-parametrization is:

$$F^2 V_{,t}{}^2\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + 1, \quad (11.2)$$

and the RLP condition resulting from it is:

$$(F_{,t} V_{,t} + FV_{,tt})\frac{\mathrm{d}t}{\mathrm{d}z}\tau + FV_{,t}\frac{\mathrm{d}\tau}{\mathrm{d}z} = 0 \quad (11.3)$$

(we ignore the cases $FV_{,t} = 0$ – prohibited in (2.1), and $\mathrm{d}t/\mathrm{d}z = 0$ – which defines a spacelike curve).

We denote:

$$H_i \stackrel{\text{def}}{=} \frac{V_{,ti}}{V_{,t}}, \qquad G_i \stackrel{\text{def}}{=} H_{i,t}, \qquad i = 1, 2, 3, \quad (11.4)$$

where $(x^1, x^2, x^3) \stackrel{\text{def}}{=} (x, y, z)$. We find $\mathrm{d}\tau/\mathrm{d}z$ from (11.3) and use it in the RLP conditions resulting from (10.9) – (10.10) in the way described above. The result is:

$$G_x - G_z\frac{\mathrm{d}x}{\mathrm{d}z} = 0, \quad (11.5)$$

$$G_y - G_z\frac{\mathrm{d}y}{\mathrm{d}z} = 0. \quad (11.6)$$

Since the analysis of (11.5) – (11.6) is complicated, we state here only the results, and present the calculations in Appendix A. The RLPs defined by (11.5) – (11.6) exist in the following cases:

(1) (Case 1.1.1.1.2.1.1 of Appendix A)

When y_0 is defined by (A23), $K \stackrel{\text{def}}{=} k/(4R)$ by (A24) and R by (A39), some of the null geodesics are RLPs, and they are solutions of (A52), with $\mathcal{F}_1$ and $\mathcal{G}_1$ given by (A40) – (A42) and (A50) – (A51). It is not known whether this subcase of the Stephani solution admits an

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[4] Similarly to what footnote #1 says, a quick way to obtain the equations that follow is to take the differentials of (10.8) – (10.10) at constant (x, y, z) and replace $\mathrm{d}t$ by τ.

FLRW limit. If it does, then only after a reparametrisation. As noted for the plane and hyperbolically symmetric cases, when k and R are parametrised by x_0, y_0 or z_0, the FLRW limit cannot be calculated.

Several subcases of this spacetime appear separately in Appendix A, but they are not listed here.

(2) (Case 1.1.1.2.1.2.1 of Appendix A)

When $y_0 = D_1 x_0$, $z_0 = C_3 x_0$, and R is determined by (A64), but $K(t) = k/(4R)$ is arbitrary, again some of the null geodesics are RLPs. They are determined by (A52), where $\mathcal{F}_1$ and $\mathcal{G}_1$ are given by (A66) – (A67).

Also here, several subcases appear separately, but they do not admit more RLPs.

(3) (Case 1.1.2.2.2 of Appendix A)

This is a copy of the subcase of the spherically symmetric Stephani solution discussed in Sec. V. All of its null geodesics are RLPs, and it contains the FLRW limit in full generality. It is not identical to FLRW because in general it has nonzero acceleration.

(4) (Case 1.2.1.2 of Appendix A)

Then $x_0 = C_1 z_0$, $y_0 = C_3 z_0$, $R(t)$ is arbitrary, V and k are determined by (A111) and (A112). This is an axially symmetric subcase of the Stephani model, and its RLPs are determined by (A113). All the RLPs intersect the symmetry axis $x = y = 0$ in each space of constant t.

(5) (Case 2.1 of Appendix A)

Then $y_0 = C_2 x_0$, $z_0 = C_3 x_0$, $K = k/(4R)$ is determined by (A137) and R is determined by (A141). As indicated there, $C_2 = C_3 = 0$ may be achieved by a coordinate transformation, and then the model is seen to be axially symmetric. All of its null geodesics are RLPs. It is less general than the previous one (see under (4) above) because the former has $R(t)$ arbitrary, while the current one has R determined by (A141).

XII. SUMMARY

The existence of repeatable light paths (RLPs) was investigated for the cosmological models found by Barnes [4] and Stephani [6]. They are called shearfree normal (SFN) because the perfect fluid source in the Einstein equations moves with zero shear and zero rotation. The Barnes models are either spherically symmetric (SS) or plane symmetric (PS) or hyperbolically symmetric (HS). In general, in each of these classes only those null geodesics are RLPs that are orthogonal to the symmetry orbits (the radial ones in the SS case). However, each one contains a subclass in which all null geodesics are RLPs. This subclass is in each case more general than FLRW. In the SS case, this special subclass has the metric (2.4) with V given by (5.9) and (5.10), and contains the whole FLRW family. In the PS case, the special subclass has V given by (7.9) – (7.13) and contains only those FLRW models for which the curvature index $k \leq 0$. In the HS case, the special subclass is given by (9.13) – (9.17) and contains only the $k < 0$ FLRW models. All these special subclasses are conformally flat, but less general than the corresponding conformally flat limits of the SS, PS and HS cases.

For the Stephani models the situation is summarised in Sec. XI. In general, no RLPs exist. The conditions of existence of RLPs, (11.5) – (11.6), put limitations on the Stephani metric. There exist subcases in which some of the null geodesics are RLPs, for example the axially symmetric subcase given by (A111) – (A112). There exists also a subclass in which all null geodesics are RLPs, it is axially symmetric as well.

This study, as explained in the introduction, was motivated by the question: what is the geometrical condition for the existence of RLPs; is it vanishing shear in the flow of the cosmic medium, as suggested by the result of Ref. [1], or rather a high symmetry of the spacetime? The solutions of Einstein's equations investigated in the present paper all have zero shear, and yet their null geodesics are RLPs only in special situations. Thus, the key to the RLP property is a symmetry of the spacetime rather than vanishing shear.

Appendix A: RLPs obeying (11.5) – (11.6)

The calculations in this Appendix are trivial in principle. The reason why they are presented in some detail is that the various separate subcases form a very complicated binary tree that would be difficult to duplicate without a guidebook.

Case 1: The general case: $G_z \neq 0$

Then (11.5) – (11.6) become

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{G_x}{G_z}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{G_y}{G_z}. \tag{A1}$$

We calculate the derivatives of $\mathrm{d}x/\mathrm{d}z$ and $\mathrm{d}y/\mathrm{d}z$ along a null geodesic by the rule

$$\frac{\mathrm{D}}{\mathrm{d}z} = \frac{\mathrm{d}t}{\mathrm{d}z}\frac{\partial}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}z}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

and substitute the results in (10.9) – (10.10). Then we use (A1) and (11.2) to eliminate $\mathrm{d}x/\mathrm{d}z$, $\mathrm{d}y/\mathrm{d}z$ and $(\mathrm{d}t/\mathrm{d}z)^2$. The resulting equations are

$$G_z \left(G_z G_{x,t} - G_x G_{z,t} \right) \frac{\mathrm{d}t}{\mathrm{d}z} + G_x \left(G_z G_{x,x} - G_x G_{z,x} \right)$$
$$+ G_y \left(G_z G_{x,y} - G_x G_{z,y} \right) + G_z \left(G_z G_{x,z} - G_x G_{z,z} \right)$$
$$+ \left(H_x G_z - H_z G_x \right) \left(G_x{}^2 + G_y{}^2 + G_z{}^2 \right) = 0, \tag{A2}$$

$$G_z \left(G_z G_{y,t} - G_y G_{z,t} \right) \frac{\mathrm{d}t}{\mathrm{d}z} + G_x \left(G_z G_{y,x} - G_y G_{z,x} \right)$$
$$+ G_y \left(G_z G_{y,y} - G_y G_{z,y} \right) + G_z \left(G_z G_{y,z} - G_y G_{z,z} \right)$$
$$+ \left(H_y G_z - H_z G_y \right) \left(G_x{}^2 + G_y{}^2 + G_z{}^2 \right) = 0. \tag{A3}$$

After (11.4) are substituted for G_i and H_i, the terms free of $\mathrm{d}t/\mathrm{d}z$ sum up to zero in each of (A2) – (A3), so both

coefficients of dt/dz must be zero, too. This implies:

$$\left(\frac{G_x}{G_z}\right)_{,t} = \left(\frac{G_y}{G_z}\right)_{,t} = 0. \qquad \text{(A4)}$$

The integrals of these are:

$$G_x = \mathcal{F}_1(x,y,z)G_z, \qquad G_y = \mathcal{G}_1(x,y,z)G_z, \qquad \text{(A5)}$$

where $\mathcal{F}_i$ and $\mathcal{G}_i$, $i = 1,2,3$ are arbitrary functions. Recalling (11.4), these are integrated again with the result:

$$\begin{aligned}
H_x &= \mathcal{F}_1 H_z + \mathcal{F}_2(x,y,z), \\
H_y &= \mathcal{G}_1 H_z + \mathcal{G}_2(x,y,z). \qquad \text{(A6)}
\end{aligned}$$

Finally, recalling the definitions of H_x and H_y from (11.4), eqs. (A6) are integrated with the result:

$$V_{,x} = \mathcal{F}_1 V_{,z} + \mathcal{F}_2 V + \mathcal{F}_3(x,y,z), \qquad \text{(A7)}$$
$$V_{,y} = \mathcal{G}_1 V_{,z} + \mathcal{G}_2 V + \mathcal{G}_3(x,y,z). \qquad \text{(A8)}$$

We introduce the following conventions:

• $C_i, D_i, c_i, d_i, E_i, i = 1,2,\dots$ will denote arbitrary constants,

• $\mathcal{F}_j, \mathcal{G}_j, j = 4,5,\dots$ will denote arbitrary functions of spatial coordinates, not necessarily of all of them.

Since the alternatives considered below will be in most cases mutually exclusive, we will re-use the same names of constants and functions with different meanings.

In every case we will follow the same scheme of reasoning. Our initial equation (IE) will be (A7) or (A8), usually multiplied by some factor ($1/2K$ or $\mathcal{F}_2^{-1}$ or $\mathcal{G}_2^{-1}$). Then, the following operations will be executed on IE, each one followed by conclusions:

1) Differentiate IE by y and t;
2) Differentiate IE by y alone;
3) Differentiate IE by x and t;
4) Differentiate IE by x alone;
5) Differentiate IE by z and t;
6) Differentiate IE by z alone.

At this stage, $(\mathcal{F}_i, \mathcal{G}_i), i = 1,2,3, K$ and R will be defined, and the corresponding Stephani model admitting RLPs (if any) will be identified. The functions $(\mathcal{F}_1, \mathcal{G}_1)$, substituted in (A1), will define the RLPs.

We will present in detail the whole 6-step procedure only for the first case considered.

It will turn out in several places along the way that the function $x_0(t)$ is in fact constant. In those cases, we will assume that $x_0 = 0$ because this result can be achieved by the coordinate transformation $x = x' + x_0$. The same is true for the pairs (y_0, y) and (z_0, z).

Case 1.1: $\mathcal{F}_2 \neq 0$

Then (A7) can be written as:

$$\frac{x - x_0}{\mathcal{F}_2} = (z - z_0)\frac{\mathcal{F}_1}{\mathcal{F}_2} + \frac{1}{2KR} \qquad \text{(A9)}$$
$$+ \frac{1}{2}\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right] + \frac{1}{2K}\frac{\mathcal{F}_3}{\mathcal{F}_2},$$

where we have introduced the symbol:

$$K \overset{\text{def}}{=} \frac{k}{4R}. \qquad \text{(A10)}$$

In writing (A9) we assumed $k \neq 0$ because $k = 0$ is the spatially flat FLRW model, in which we know that all null geodesics are RLPs. Taking the second derivative of (A9) by y and t we obtain:

$$x_{0,t}\left(\frac{1}{\mathcal{F}_2}\right)_{,y} = z_{0,t}\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right)_{,y} + y_{0,t} - \left(\frac{1}{2K}\right)_{,t}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y}. \qquad \text{(A11)}$$

Case 1.1.1: $x_{0,t} \neq 0$

Then we divide (A11) by $x_{0,t}$ and differentiate the result by t, obtaining:

$$\left(\frac{z_{0,t}}{x_{0,t}}\right)_{,t}\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right)_{,y} + \left(\frac{y_{0,t}}{x_{0,t}}\right)_{,t}$$
$$- \left[\frac{1}{x_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = 0. \qquad \text{(A12)}$$

Case 1.1.1.1: $(z_{0,t}/x_{0,t})_{,t} \neq 0$

Then we divide (A12) by $(z_{0,t}/x_{0,t})_{,t}$ and differentiate the result by t. We get:

$$\left[\left(\frac{y_{0,t}}{x_{0,t}}\right)_{,t} \bigg/ \left(\frac{z_{0,t}}{x_{0,t}}\right)_{,t}\right]_{,t} - \chi(t)\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = 0, \qquad \text{(A13)}$$

where:

$$\chi(t) \overset{\text{def}}{=} \left\{\left[\frac{1}{x_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t} \bigg/ \left(\frac{z_{0,t}}{x_{0,t}}\right)_{,t}\right\}_{,t}. \qquad \text{(A14)}$$

Case 1.1.1.1.1: $\chi(t) \neq 0$

Then we get:

$$\frac{1}{\chi}\left[\left(\frac{y_{0,t}}{x_{0,t}}\right)_{,t} \bigg/ \left(\frac{z_{0,t}}{x_{0,t}}\right)_{,t}\right]_{,t} = \left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y}$$
$$= C_1 = \text{constant}, \qquad \text{(A15)}$$

both expressions being constant because the first one depends only on t, while the second one depends only on (x,y,z). Integrating both equations we obtain:

$$\frac{C_1}{2K} = D_1 x_0 + y_0 + D_3 z_0 + D_4, \qquad \text{(A16)}$$
$$\mathcal{F}_3 = (C_1 y + \mathcal{F}_4(x,z))\mathcal{F}_2. \qquad \text{(A17)}$$

In principle, we would have to consider the cases $C_1 \neq 0$ and $C_1 = 0$ separately. However, they lead to the same result. When $C_1 \neq 0$, (A16) determines $1/(2K)$. Then we substitute (A16) and (A17) in (A12) and obtain

$$\left(\frac{z_{0,t}}{x_{0,t}}\right)_{,t}\left[\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right)_{,y} - D_3\right] = 0. \qquad \text{(A18)}$$

Since in Case 1.1.1.1 the first factor is nonzero, we have:

$$\mathcal{F}_1 = (D_3 y + \mathcal{F}_5(x, z)) \, \mathcal{F}_2. \tag{A19}$$

When we substitute (A17) and (A19) in (A11) we obtain:

$$x_{0,t} \left[\left(\frac{1}{\mathcal{F}_2} \right)_{,y} + D_1 \right] = 0. \tag{A20}$$

Since $x_{0,t} \neq 0$ in our current Case 1.1.1, we have:

$$\frac{1}{\mathcal{F}_2} = -D_1 y + \mathcal{F}_6(x, z). \tag{A21}$$

Using (A21), (A19) and (A17) in (A9) we obtain:

$$x (-D_1 y + \mathcal{F}_6) - x_0 \mathcal{F}_6 = z (D_3 y + \mathcal{F}_5) - z_0 \mathcal{F}_5$$
$$+ \frac{1}{2KR} + \frac{1}{2} \left[(x - x_0)^2 + y^2 + y_0{}^2 + (z - z_0)^2 \right]$$
$$+ \frac{1}{2K} \mathcal{F}_4 + D_4 y. \tag{A22}$$

The derivative of this by y now gives $-D_1 x = D_3 z + y + D_4$, which is a clear contradiction.

When $C_1 = 0$, K remains undetermined, and instead (A15) determines $y_0 = -D_1 x_0 - D_3 z_0$. Equations (A19) and (A21) still follow and (A22) results with $D_4 = 0$, leading to the same contradiction.

Case 1.1.1.1.1 thus turned out to be empty, and we go back to (A13) to consider:

Case 1.1.1.1.2: $\chi(t) = 0$

Then (A13) implies $[(y_{0,t}/x_{0,t})_{,t} / (z_{0,t}/x_{0,t})_{,t}]_{,t} = 0$, which is integrated with the result:

$$y_0 = C_1 x_0 + C_3 z_0 \tag{A23}$$

(the additive constant was set to zero by a transformation $y = y' +$ constant), and $\chi(t) = 0$ implies, via (A14)

$$\frac{1}{2K} = D_1 x_0 + D_3 z_0 + D_4. \tag{A24}$$

We substitute (A23) – (A24) in (A12) and obtain

$$\left(\frac{z_{0,t}}{x_{0,t}} \right)_{,t} \left[\left(\frac{\mathcal{F}_1}{\mathcal{F}_2} \right)_{,y} - D_3 \left(\frac{\mathcal{F}_3}{\mathcal{F}_2} \right)_{,y} + C_3 \right] = 0. \tag{A25}$$

Since we are still in Case 1.1.1.1, where $(z_{0,t}/x_{0,t})_{,t} \neq 0$, the above leads to

$$\mathcal{F}_1 = D_3 \mathcal{F}_3 + (\mathcal{F}_4(x, z) - C_3 y) \, \mathcal{F}_2. \tag{A26}$$

Substituting this and (A23) – (A24) in (A11) we get

$$x_{0,t} \left[\left(\frac{1}{\mathcal{F}_2} \right)_{,y} - C_1 + D_1 \left(\frac{\mathcal{F}_3}{\mathcal{F}_2} \right)_{,y} \right] = 0. \tag{A27}$$

In Case 1.1.1, where $x_{0,t} \neq 0$, the above implies

$$D_1 \mathcal{F}_3 = (C_1 y + \mathcal{F}_5(x, z)) \, \mathcal{F}_2 - 1. \tag{A28}$$

Case 1.1.1.1.2.1: $D_1 \neq 0$

Then (A28) determines $\mathcal{F}_3$. Using (A26), (A24) and (A23) in (A9) we get:

$$\frac{x}{\mathcal{F}_2} = z \left(D_3 \frac{\mathcal{F}_3}{\mathcal{F}_2} + \mathcal{F}_4 - C_3 y \right) - z_0 \mathcal{F}_4 + \frac{1}{2KR}$$
$$+ \frac{1}{2} \left[(x - x_0)^2 + y^2 + (C_1 x_0 + C_3 z_0)^2 + \right.$$
$$\left. (z - z_0)^2 \right] + D_4 \frac{\mathcal{F}_3}{\mathcal{F}_2} + x_0 \mathcal{F}_5. \tag{A29}$$

We differentiate this by y alone and obtain

$$x \left(\frac{1}{\mathcal{F}_2} \right)_{,y} = z \left[D_3 \left(\frac{\mathcal{F}_3}{\mathcal{F}_2} \right)_{,y} - C_3 \right] + y + D_4 \left(\frac{\mathcal{F}_3}{\mathcal{F}_2} \right)_{,y}. \tag{A30}$$

Integrating this back and using (A28) for $\mathcal{F}_3$ we obtain

$$\frac{1}{\mathcal{F}_2} = \frac{\frac{1}{2} D_1 y^2 + (C_1 D_3 - C_3 D_1) \, yz + C_1 D_4 y}{D_1 x + D_3 z + D_4} + \mathcal{F}_6(x, z). \tag{A31}$$

Using (A28) and (A31) in (A29) we obtain

$$x \mathcal{F}_6(x, z) = (z - z_0) \, \mathcal{F}_4(x, z) + \frac{1}{2KR} + x_0 \mathcal{F}_5(x, z)$$
$$+ \frac{1}{2} \left[(x - x_0)^2 + (C_1 x_0 + C_3 z_0)^2 + (z - z_0)^2 \right]$$
$$+ \frac{1}{D_1} (D_3 z + D_4) (\mathcal{F}_5(x, z) - \mathcal{F}_6(x, z)). \tag{A32}$$

Taking the second derivative of this by x and t we obtain

$$- \frac{z_{0,t}}{x_{0,t}} \mathcal{F}_{4,x} - 1 + \mathcal{F}_{5,x} = 0. \tag{A33}$$

In Case 1.1.1.1, where $(z_{0,t}/x_{0,t})_{,t} \neq 0$, this implies

$$\mathcal{F}_4 = \mathcal{F}_4(z), \qquad \mathcal{F}_5 = x + \mathcal{F}_7(z). \tag{A34}$$

Putting this in (A32) and taking its derivative by x alone, then integrating back, we get

$$\left(x + \frac{D_3 z + D_4}{D_1} \right) \mathcal{F}_6(x, z) = \frac{1}{2} x^2 + \frac{x (D_3 z + D_4)}{D_1} + \mathcal{F}_8(z). \tag{A35}$$

Using this in (A32), then taking the second derivative by z and t we get

$$- \frac{z_{0,t}}{x_{0,t}} (\mathcal{F}_{4,z} + 1) + \mathcal{F}_{7,z} = 0. \tag{A36}$$

In the present case this means

$$\mathcal{F}_7 = C_4 = \text{constant}, \qquad \mathcal{F}_4 = -z + C_5. \tag{A37}$$

Feeding this information in (A32) and using (A24) we get

$$\mathcal{F}_8(z) - \frac{D_3 C_4}{D_1} z + \frac{1}{2} z^2 - C_5 z - \frac{D_4}{D_1} C_4 = E_1, \tag{A38}$$
$$\frac{1}{R} (D_1 x_0 + D_3 z_0 + D_4) + C_4 x_0 - C_5 z_0$$

$$+\frac{1}{2}\left[x_0{}^2 + (C_1 x_0 + C_3 z_0)^2 + z_0{}^2\right] = E_1, \qquad \text{(A39)}$$

the two expressions being constant because they are equal, while the first one depends only on z and the second one only on t. Equations (A38) – (A39) define $R(t)$ and $\mathcal{F}_8(z)$, and are the final solution of (A7).

Picking up the pieces, we obtain the following formula:

$$\mathcal{F}_1 = \frac{\mathcal{U}_1}{\mathcal{U}_2}, \qquad \text{(A40)}$$

$$\mathcal{U}_1 \overset{\text{def}}{=} \frac{1}{2}D_3\left(x^2 - y^2 - z^2\right) + (C_1 D_3 - C_3 D_1)\, xy$$
$$-D_1 xz + (C_4 D_3 + C_5 D_1)\, x - C_3 D_4 y - D_4 z$$
$$+C_5 D_4 - D_3 E_1, \qquad \text{(A41)}$$

$$\mathcal{U}_2 \overset{\text{def}}{=} \frac{1}{2}D_1\left(x^2 + y^2 - z^2\right) + D_3 xz$$
$$+ (C_1 D_3 - C_3 D_1)\, yz + D_4 x + C_1 D_4 y$$
$$+ (C_4 D_3 + C_5 D_1)\, z + C_4 D_4 + D_1 E_1. \quad \text{(A42)}$$

Now we will deal with (A8), still within Case 1.1.1.1.2.1.

Case 1.1.1.1.2.1.1: $\mathcal{G}_2 \neq 0$

Then (A8) is written as

$$\frac{y - C_1 x_0 - C_3 z_0}{\mathcal{G}_2} = (z - z_0)\frac{\mathcal{G}_1}{\mathcal{G}_2} + \frac{1}{2KR} + \frac{1}{2K}\frac{\mathcal{G}_3}{\mathcal{G}_2}$$
$$+\frac{1}{2}\left[(x - x_0)^2 + (y - C_1 x_0 - C_3 z_0)^2 + (z - z_0)^2\right],$$
$$\text{(A43)}$$

We follow exactly the same sequence of steps that we did in solving (A9). We take the second derivative of (A43) by y and t and obtain, using (A24):

$$\left(C_1 + C_3\frac{z_{0,t}}{x_{0,t}}\right)\left(\frac{1}{\mathcal{G}_2}\right)_{,y} = \frac{z_{0,t}}{x_{0,t}}\left(\frac{\mathcal{G}_1}{\mathcal{G}_2}\right)_{,y} \qquad \text{(A44)}$$
$$+\left(C_1 + C_3\frac{z_{0,t}}{x_{0,t}}\right) - \left(D_1 + D_3\frac{z_{0,t}}{x_{0,t}}\right)\left(\frac{\mathcal{G}_3}{\mathcal{G}_2}\right)_{,y}.$$

Since we are in Case 1.1.1.1, the coefficients of $(z_{0,t}/x_{0,t})$ and the remaining terms must balance separately. Moreover, we are in Case 1.1.1.1.2.1, where $D_1 \neq 0$. Integrating the two equations with respect to y we obtain

$$\frac{\mathcal{G}_1}{\mathcal{G}_2} = \left(\frac{C_1 D_3}{D_1} - C_3\right)\left(y - \frac{1}{\mathcal{G}_2}\right) + \mathcal{G}_4(x, z),$$
$$+ D_3 \mathcal{G}_5(x, z), \qquad \text{(A45)}$$

$$\frac{\mathcal{G}_3}{\mathcal{G}_2} = \frac{C_1}{D_1}\left(y - \frac{1}{\mathcal{G}_2}\right) + \mathcal{G}_5(x, z). \qquad \text{(A46)}$$

We substitute (A45) – (A46) in (A43) differentiated by y alone and obtain

$$\frac{1}{\mathcal{G}_2} + y\left(\frac{1}{\mathcal{G}_2}\right)_{,y} = z\left(\frac{C_1 D_3}{D_1} - C_3\right)\left[1 - \left(\frac{1}{\mathcal{G}_2}\right)_{,y}\right]$$

$$+\frac{C_1 D_4}{D_1}\left[1 - \left(\frac{1}{\mathcal{G}_2}\right)_{,y}\right] + y. \qquad \text{(A47)}$$

From here:

$$\frac{1}{\mathcal{G}_2} = \qquad\qquad\qquad\qquad\qquad \text{(A48)}$$
$$\frac{\frac{1}{2}D_1 y^2 + (C_1 D_3 - C_3 D_1)\, yz + C_1 D_4 y + D_1 \mathcal{G}_6(x, z)}{D_1 y + (C_1 D_3 - C_3 D_1)\, z + C_1 D_4}.$$

After substituting (A45), (A46), (A48), (A23) and (A24) in (A43) we obtain

$$(z - z_0)\,\mathcal{G}_4(x, z) + (D_1 x_0 + D_3 z_0 + D_4)\frac{1}{R}$$
$$+\frac{1}{2}\left[(x - x_0)^2 + (C_1 x_0 + C_3 z_0)^2 + (z - z_0)^2\right]$$
$$+ (D_1 x_0 + D_3 z + D_4)\,\mathcal{G}_5(x, z) - \mathcal{G}_6(x, z) = 0. \quad \text{(A49)}$$

Following further exactly the same scheme that we presented in solving (A7) we arrive at an equation determining $R(t)$ that is identical with (A39), i.e. does not put any additional limitation on the R determined by (A39). Thus, (A39) gives the final condition for the existence of RLPs in Case 1.1.1.1.2.1.

Putting together all the partial results we get

$$\mathcal{G}_1 = \frac{\mathcal{U}_3}{\mathcal{U}_2}, \qquad \text{(A50)}$$

$$\mathcal{U}_3 \overset{\text{def}}{=} \frac{1}{2}(C_1 D_3 - C_3 D_1)\left(-x^2 + y^2 - z^2\right)$$
$$+ D_3 xy - D_1 yz + C_3 D_4 x + (C_4 D_3 + C_5 D_1)\, y$$
$$- C_1 D_4 z + D_4(C_1 C_5 + C_3 C_4)$$
$$- E_1(C_1 D_3 - C_3 D_1), \qquad \text{(A51)}$$

where $\mathcal{U}_2$ is given in (A42). Now the equations

$$\frac{dx}{dz} = \mathcal{F}_1(x, y, z), \qquad \frac{dy}{dz} = \mathcal{G}_1(x, y, z) \qquad \text{(A52)}$$

do determine the RLPs, with $\mathcal{F}_1(x, y, z)$ and $\mathcal{G}_1(x, y, z)$ given by (A40) – (A42) and (A50) – (A51).[5]

By comparing (A24) and (A39) with (7.15) – (7.16) we see that the latter is contained in (A39) as the subcase $x_0 = 0 = C_3$, with D_3, D_4, C_5 and E_1 expressed in terms of C, A_1, A_2, B_1 and B_2.

Likewise, comparing (A24) and (A39) with (9.19) – (9.20) we see that the latter is contained in (A39) as the subcase $x_0 = 0 = C_3 = C_5$, with y_0 transformed to z_0 by the coordinate transformation $(y, z) = (z', y')$. The other constants in (A39) are expressed in terms of those from (9.19) – (9.20).

These two special Stephani solutions will be contained also in several other subcases of $\{(A24), (A39)\}$ that will

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[5] The correctness of (A39) – (A42) and (A50) – (A51) was verified by the computer algebra program Ortocartan [12, 13].

appear in the following text. The spherically symmetric Stephani solution will appear in several places as well, but each time it will be easy to recognise.

For completeness, we will still consider:

Case 1.1.1.1.2.1.2: $\mathcal{G}_2 = 0$

With $D_1 \neq 0 \neq x_{0,t}$ and $(z_{0,t}/x_{0,t})_{,t} \neq 0$ still applying, $\mathcal{G}_2 = 0$ quickly leads to a contradiction; it is enough to divide (A8) by $2K$, then differentiate the result by y and t and use the conclusion in the derivative by y alone. This case is thus empty.

Case 1.1.1.1.2.1 is thereby exhausted, and we go back to (A28) to consider:

Case 1.1.1.1.2.2: $D_1 = 0$

By applying the same scheme as with $D_1 \neq 0$ we obtain formulae for $\mathcal{F}_1$, $\mathcal{G}_1$ and R that are the subcases $D_1 = 0$ of (A39), (A40) – (A42) and (A50) – (A51).

When $C_1 = 0$, we obtain $x_{0,t} = 0$, which contradicts the definition of Case 1.1.1.

This exhausts Case 1.1.1.1, so we go back to (A12) to consider

Case 1.1.1.2: $(z_{0,t}/x_{0,t})_{,t} = 0$

Then (A12) becomes

$$\left(\frac{y_{0,t}}{x_{0,t}}\right)_{,t} - \left[\frac{1}{x_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = 0. \quad \text{(A53)}$$

Case 1.1.1.2.1: $[(1/x_{0,t})(1/2K)_{,t}]_{,t} \neq 0$

For the sake of comparison with the previously considered cases, from this place up to (A65) we replace the capital letters denoting constants by their corresponding l.c. letters. Then the solution of (A53) is

$$\left(\frac{y_{0,t}}{x_{0,t}}\right)_{,t} \bigg/ \left[\frac{1}{x_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t} = \left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = c_1. \quad \text{(A54)}$$

Case 1.1.1.2.1.1: $c_1 \neq 0$

We then get

$$\frac{c_1}{2K} = d_1 x_0 + y_0 + d_4, \quad \text{(A55)}$$

$$\mathcal{F}_3 = (c_1 y + \mathcal{F}_4(x,z))\,\mathcal{F}_2. \quad \text{(A56)}$$

$$z_0 = c_3 x_0, \quad \text{(A57)}$$

the last one from the definition of Case 1.1.1.2.

We substitute (A55) – (A57) in (A11). In Case 1.1.1, where $x_{0,t} \neq 0$, the resulting equation integrates as

$$c_3 \mathcal{F}_1 = 1 + (d_1 y - \mathcal{F}_5(x,z))\,\mathcal{F}_2. \quad \text{(A58)}$$

Case 1.1.1.2.1.1.1: $c_3 \neq 0$

We then use (A55) – (A58) in (A9) differentiated by y and obtain

$$\frac{1}{\mathcal{F}_2} = \frac{\frac{1}{2}y^2 + \frac{d_1}{c_3}yz + d_4 y + \mathcal{F}_6(x,z)}{x - z/c_3}. \quad \text{(A59)}$$

After substituting (A55) – (A59) in (A9) we obtain

$$-\frac{z}{c_3}\mathcal{F}_5(x,z) + \frac{1}{2KR} + x_0\mathcal{F}_5(x,z) - \mathcal{F}_6(x,z)$$
$$+\frac{1}{2}\left[(x-x_0)^2 + y_0{}^2 + (z-c_3 x_0)^2\right]$$
$$+\frac{1}{c_1}(d_1 x_0 + y_0 + d_4)\,\mathcal{F}_4(x,z) = 0. \quad \text{(A60)}$$

Following the same sequence of steps that we described below (A8), we arrive at:

$$\frac{1}{c_1 R}(d_1 x_0 + y_0 + d_4) + \frac{1}{2}\left[(1+c_3{}^2)x_0{}^2 + y_0{}^2\right]$$
$$+c_5 x_0 + c_6 y_0 = E_1 = \text{constant}. \quad \text{(A61)}$$

This is equivalent to the subcase $C_3 = 0$ of (A39) transformed by $(y,z) = (z',y')$ (this transformation interchanges the names of y_0 and z_0). Note from (A5) that the interchange of y and z implies the interchange of G_y and G_z, and consequently the transformation $(\mathcal{F}_1, \mathcal{G}_1) \to (\mathcal{F}_1/\mathcal{G}_1, 1/\mathcal{G}_1)$. Thus the $\mathcal{F}_1$ and $\mathcal{G}_1$ for (A61) are obtained from (A40) – (A42) and (A50) – (A51) in this way, with the substitutions $C_3 = 0$ and $(y,z) \to (z,y)$. To have a consistent naming of the constants, one should do the following replacement in (A61):

$$(d_1/c_1, 1/c_1, d_4/c_1, c_3, c_5, c_6) = (D_1, D_3, D_4, C_1, C_4, -C_5). \quad \text{(A62)}$$

This is the end of Case 1.1.1.2.1.1.1. We go back to (A58) and consider now

Case 1.1.1.2.1.1.2: $c_3 = 0$

Even though the limit $c_3 = 0$ is singular in (A59)–(A60), by going through the usual procedure we end up with the subcase $c_3 = 0$ of (A61). This is also a regular subcase of the current $\mathcal{F}_1$ and $\mathcal{G}_1$, calculated in the way explained above.

Thus we go back to (A54) and consider now

Case 1.1.1.2.1.2: $c_1 = 0$

Then (A55) does not exist because (A54) only determines $(y_{0,t}/x_{0,t})_{,t} = 0$ and $(\mathcal{F}_3/\mathcal{F}_2)_{,y} = 0$. From the first of these we have

$$y_0 = C_2 x_0, \quad \text{(A63)}$$

and (A56) still holds with $c_1 = 0$.

Case 1.1.1.2.1.2.1: $c_3 \neq 0$

Proceeding as before by consecutive differentiations we end up with the following equation:

$$\frac{1}{2KR} + \frac{c_4}{2K} + \frac{1}{2}\left(1 + C_2{}^2 + c_3{}^2\right)x_0{}^2 + c_5 x_0 + c_6 = 0. \quad \text{(A64)}$$

The function $K(t)$ is still arbitrary at this point. Since in the acceleration-free limit x_0 and $KR = k/4$ are constant, we see that $k/R = \text{constant}$ in this limit. So, (A64) can reproduce the $k = 0$ FLRW limit.

Running this solution through (A8) with $\mathcal{G}_2 \neq 0$, with the current values of y_0 and z_0, and with $[(1/x_{0,t})(1/2K)_{,t}]_{,t} \neq 0$ as appropriate for the current Case 1.1.1.2.1, we obtain

$$\frac{1}{2KR} + \frac{E_1}{2K} + \frac{1}{2}\left(1 + C_2{}^2 + c_3{}^2\right)x_0{}^2 + E_2 x_0 + E_3 = 0, \tag{A65}$$

where E_i, $i = 1, 2, 3$, are new arbitrary constants. Choosing $(E_1, E_2, E_3) = (c_4, c_5, c_6)$ we make (A65) identical to (A61). Then (A64) does define a subcase of the Stephani model, different from (A39), that also has RLPs. The corresponding $\mathcal{F}_1$ and $\mathcal{G}_1$ are found to be

$$\mathcal{F}_1 = \frac{\frac{1}{2}\left(x^2 - y^2 - z^2\right) + C_2 xy + C_3 xz + C_5 x + C_6}{\frac{1}{2}C_3\left(-x^2 - y^2 + z^2\right) + xz + C_2 yz + C_5 z + C_3 C_6}, \tag{A66}$$

$$\mathcal{G}_1 = \frac{\frac{1}{2}C_2\left(x^2 - y^2 + z^2\right) - xy - C_3 yz - C_5 y - C_2 C_6}{\frac{1}{2}C_3\left(x^2 + y^2 - z^2\right) - xz - C_2 yz - C_5 z - C_3 C_6}, \tag{A67}$$

Even though the subcase $C_3 = 0$ requires separate treatment at intermediate stages of the calculation, the final formulae for $1/R$, $\mathcal{F}_1$ and $\mathcal{G}_1$ turn out to be contained in (A64), (A66) and (A67) as the regular subcase $C_3 = 0$. While considering $C_3 = 0$, the subcase $C_2 = 0$ also requires separate treatment, but in the end leads to R, $\mathcal{F}_1$ and $\mathcal{G}_1$ given by (A64) – (A67) with $C_3 = C_2 = 0$. When $C_2 = 0$, the $\mathcal{G}_2$ in (A8) must be zero.[6]

When we run the general expression for (A64) through (A8) with $\mathcal{G}_2 = 0$, we quickly obtain $c_3 = C_2 = 0$ as a necessary consequence.

Having thus exhausted Case 1.1.1.2.1 we go back to (A53) and consider

Case 1.1.1.2.2: $[(1/x_{0,t})(1/2K)_{,t}]_{,t} = 0$

Then, using (A53) and the definitions of Cases 1.1.1.2 and 1.1.1.2.2 we obtain:

$$\frac{1}{2K} = D_1 x_0 + D_2, \qquad y_0 = C_2 x_0, \qquad z_0 = C_3 x_0. \tag{A68}$$

Proceeding further by the ordinary scheme we obtain from (A9):

$$\frac{1}{R}\left(D_1 x_0 + D_2\right) + \frac{1}{2}\left(1 + C_2{}^2 + C_3{}^2\right)x_0{}^2 + C_4 x_0 = C_5. \tag{A69}$$

This is formally a subcase of (A39), but it has $G_x = G_y = G_z = 0$, and so belongs in Case 2 considered further on.

This exhausts Case 1.1.1, so we go back to (A11) and consider

Case 1.1.2: $x_{0,t} = 0$

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[6] The correctness of (A64) – (A67) was verified by the computer algebra program Ortocartan [12, 13].

Then (A11) becomes

$$z_{0,t}\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right)_{,y} + y_{0,t} - \left(\frac{1}{2K}\right)_{,t}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = 0. \tag{A70}$$

Case 1.1.2.1: $y_{0,t} \neq 0$

Then we divide (A70) by $y_{0,t}$ and differentiate the result by t, obtaining

$$\left(\frac{z_{0,t}}{y_{0,t}}\right)_{,t}\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right)_{,y} - \left[\frac{1}{y_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = 0. \tag{A71}$$

Case 1.1.2.1.1: $(z_{0,t}/y_{0,t})_{,t} \neq 0$

Then we divide (A71) by $(z_{0,t}/y_{0,t})_{,t}$ and differentiate the result by t.

Case 1.1.2.1.1.1: $\chi_1(t) \neq 0$

where

$$\chi_1(t) \stackrel{\text{def}}{=} \left\{\left[\frac{1}{y_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t} \middle/ \left(\frac{z_{0,t}}{y_{0,t}}\right)_{,t}\right\}_{,t}. \tag{A72}$$

Then $(\mathcal{F}_3/\mathcal{F}_2)_{,y} = 0$ and (A71) immediately implies $(\mathcal{F}_1/\mathcal{F}_2)_{,y} = 0$. However, then (A70) gives a contradiction with $y_{0,t} \neq 0$. This case is thus empty, so we proceed to consider the complementary

Case 1.1.2.1.1.2: $\chi_1(t) = 0$

Then we have

$$\frac{1}{2K} = D_1 + D_2 y_0 + D_3 z_0, \tag{A73}$$

and from (A71)

$$\frac{\mathcal{F}_1}{\mathcal{F}_2} = D_3 \frac{\mathcal{F}_3}{\mathcal{F}_2} + \mathcal{F}_4(x, z). \tag{A74}$$

Proceeding from (A9) by the usual method we arrive at

$$\frac{1}{R}\left(D_1 + D_2 y_0 + D_3 z_0\right) + \frac{1}{2}\left(y_0{}^2 + z_0{}^2\right)$$
$$+ C_4 y_0 + C_5 z_0 = E_1. \tag{A75}$$

This is equivalent to the subcase $C_1 = C_3 = 0$ of (A39) under the coordinate transformation $(x, y) = (y', x')$ that interchanges the names of x_0 and y_0. Note, by looking at (A7) – (A8), that the interchange of x and y implies the interchange of $\mathcal{F}_1$ and $\mathcal{G}_1$. Thus, the corresponding $\mathcal{F}_1$ and $\mathcal{G}_1$ are found from (A40) – (A42) and (A50) – (A51) as, respectively, the old $\mathcal{G}_1$ and $\mathcal{F}_1$ in the limit $C_1 = C_3 = 0$ with $(x, y) \to (y', x')$, and with the additional renaming $(C_5, D_1, D_4) \to (-C_5, D_2, D_1)$ (compare (A39) with (A75)).

So we go back to (A71) to consider

Case 1.1.2.1.2: $(z_{0,t}/y_{0,t})_{,t} = 0$

Then

$$z_0 = C_3 y_0. \qquad (A76)$$

Case 1.1.2.1.2.1: $[(1/y_{0,t})(1/2K)_{,t}]_{,t} \neq 0$

Then (A71) gives

$$\mathcal{F}_3/\mathcal{F}_2 = \mathcal{F}_4(x, z), \qquad (A77)$$

and (A70) implies

$$C_3 \frac{\mathcal{F}_1}{\mathcal{F}_2} = -y + \mathcal{F}_5(x, z). \qquad (A78)$$

Note that $C_3 \neq 0$, or else (A78) is a contradiction.

Proceeding from (A9) by the usual routine we obtain

$$\frac{1}{2KR} + \frac{C_4}{2K} + \frac{1}{2}\left(1 + C_3{}^2\right) y_0{}^2 + C_5 y_0 + C_6 = 0, \ (A79)$$

with $K(t)$ undetermined. This is equivalent to the subcase $C_2 = 0$ of (A64). The transformation from (A64) and (A66) – (A67) to the current case is $(x, y) = (y', x')$, with the accompanying renaming $(x_0, y_0) \rightarrow (y_0, x_0)$. As explained under (A75), the interchange of x and y implies the interchange of $\mathcal{F}_1$ and $\mathcal{G}_1$, so the current $\mathcal{F}_1$ and $\mathcal{G}_1$ are obtained as the $\mathcal{G}_1$ and $\mathcal{F}_1$, respectively, of (A66) – (A67) with $(x, y) \rightarrow (y, x)$ and $C_2 = 0$.

So, we go back to (A71) once more and consider the second possibility:

Case 1.1.2.1.2.2: $[(1/y_{0,t})(1/2K)_{,t}]_{,t} = 0$

Then

$$\frac{1}{2K} = D_1 + D_2 y_0, \qquad (A80)$$

and from (A70), since $y_{0,t} \neq 0$,

$$C_3 \frac{\mathcal{F}_1}{\mathcal{F}_2} = D_2 \frac{\mathcal{F}_3}{\mathcal{F}_2} - y + \mathcal{F}_4(x, z). \qquad (A81)$$

Case 1.1.2.1.2.2.1: $C_3 \neq 0$

Then, by the usual routine, (A81) used in (A9) leads to

$$\frac{1}{R}\left(D_1 + D_2 y_0\right) + \frac{1}{2}\left(1 + C_3{}^2\right) y_0{}^2 + C_4 y_0 = C_5, \ (A82)$$

which is formally the subcase $C_5 = D_3 = 0$, $z_0 = C_3 x_0$ of (A75). However, in this case we have $G_x = G_y = G_z = 0$ in (11.5) – (11.6). Thus, it is in fact excluded from the present consideration and will appear later, when we consider Case 2.

Case 1.1.2.1.2.2.2: $C_3 = 0$

Then the subcase $C_3 = 0$ of (A82) is obtained, so it belongs in Case 2, too.

This exhausts Case 1.1.2.1, so we go back to (A70) to consider

Case 1.1.2.2: $y_{0,t} = 0$

Then $y_0 = 0$ by a transformation of y, and in (A70) we consider

Case 1.1.2.2.1: $z_{0,t} \neq 0$

Then (A70) may be written as

$$\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right)_{,y} - \frac{1}{z_{0,t}}\left(\frac{1}{2K}\right)_{,t}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2}\right)_{,y} = 0. \qquad (A83)$$

When $\{(1/z_{0,t})[1/(2K)]_{,t}\}_{,t} \neq 0$, this yields $(\mathcal{F}_1/\mathcal{F}_2)_{,y} = (\mathcal{F}_3/\mathcal{F}_2)_{,y} = 0$, and the usual routine leads from (A9) to

$$\frac{1}{2KR} + \frac{C_4}{2K} + \frac{1}{2}z_0{}^2 + C_5 z_0 + C_6 = 0, \qquad (A84)$$

with $K(t)$ undetermined. This is the subcase $c_3 = C_2 = 0$ of (A64) obtained by the coordinate transformation $(x, z) = (z', x')$ that interchanges x_0 with z_0. Note from (A5) that the interchange of x and z implies the interchange of G_x and G_z, and consequently the transformation $(\mathcal{F}_1, \mathcal{G}_1) \rightarrow (1/\mathcal{F}_1, \mathcal{G}_1/\mathcal{F}_1)$. Thus, the $\mathcal{F}_1$ and $\mathcal{G}_1$ for (A84) are obtained from (A66) – (A67) in this way, with the substitutions $C_2 = C_3 = 0$ and $(x, z) \rightarrow (z, x)$.

Thus we assume $\left[\frac{1}{z_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t} = 0$ in (A83). Then

$$\frac{1}{2K} = D_1 + D_3 z_0, \qquad (A85)$$

$$\frac{\mathcal{F}_1}{\mathcal{F}_2} = D_3 \frac{\mathcal{F}_3}{\mathcal{F}_2} + \mathcal{F}_4(x, z). \qquad (A86)$$

This leads from (A9) to

$$\frac{1}{R}\left(D_1 + D_3 z_0\right) + \frac{1}{2}z_0{}^2 + C_5 z_0 = E_1. \qquad (A87)$$

This is the subcase $y_0 = 0$ of (A75). As with (A82), in this case we have $G_x = G_y = G_z = 0$, and so it is excluded from the present consideration – it will appear in Case 2.

So we go back to (A71) and consider

Case 1.1.2.2.2: $z_{0,t} = 0$

With $x_{0,t} = y_{0,t} = z_{0,t} = 0$ now being the case, we are in the spherically symmetric subcase of the Stephani solution. Then (A9) may be written as

$$\frac{x}{\mathcal{F}_2} = z\frac{\mathcal{F}_1}{\mathcal{F}_2} + \frac{1}{2KR} + \frac{1}{2}\left(x^2 + y^2 + z^2\right) + \frac{1}{2K}\frac{\mathcal{F}_3}{\mathcal{F}_2}. \ (A88)$$

The t-derivative of this is

$$\left(\frac{1}{2KR}\right)_{,t} + \left(\frac{1}{2K}\right)_{,t}\frac{\mathcal{F}_3}{\mathcal{F}_2} = 0. \qquad (A89)$$

If $(2KR)_{,t} = 0$, then this is simply the FLRW family of models that we need not investigate. Consequently, we

take $(2KR)_{,t} \neq 0$. Then both the other factors must be nonzero, and

$$\left(\frac{1}{2KR}\right)_{,t} \Big/ \left(\frac{1}{2K}\right)_{,t} = -\frac{\mathcal{F}_3}{\mathcal{F}_2} = C_1. \qquad (A90)$$

This is the subcase of the spherically symmetric Stephani solution that we identified in Sec. V, and the equation above is consistent with (5.10). All of its null geodesics are RLPs. The FLRW limit of this model is $C_1 = 0$, and it includes the whole FLRW family.

Equation (A90) is the final solution of (A7). To prevent (A8) from imposing any limitations on it, it is sufficient to choose $\mathcal{G}_2 = \mathcal{G}_3 = 0$, $\mathcal{G}_1 = y/z$.

This exhausts Case 1.1, so we go back to (A7) and take

Case 1.2: $\mathcal{F}_2 = 0$

Then (A7) can be rewritten as

$$x - x_0 = (z - z_0)\,\mathcal{F}_1 + \frac{1}{2K}\mathcal{F}_3. \qquad (A91)$$

We differentiate this by y and t and obtain

$$-z_{0,t}\mathcal{F}_{1,y} + \left(\frac{1}{2K}\right)_{,t}\mathcal{F}_{3,y} = 0. \qquad (A92)$$

Case 1.2.1: $z_{0,t} \neq 0$

Then from (A91)

$$-\mathcal{F}_{1,y} + \frac{1}{z_{0,t}}\left(\frac{1}{2K}\right)_{,t}\mathcal{F}_{3,y} = 0. \qquad (A93)$$

When $\left[\frac{1}{z_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t} \neq 0$, we have $\mathcal{F}_{1,y} = \mathcal{F}_{3,y} = 0$, and then we differentiate (A91) by x to get

$$1 = (z - z_0)\,\mathcal{F}_{1,x} + \frac{1}{2K}\mathcal{F}_{3,x}. \qquad (A94)$$

Differentiating this by t we get $\mathcal{F}_{1,x} = \mathcal{F}_{3,x} = 0$ in consequence of the assumptions about the functions of t, but this is a contradiction with (A94). This means that $\left[\frac{1}{z_{0,t}}\left(\frac{1}{2K}\right)_{,t}\right]_{,t} = 0$, i.e.

$$\frac{1}{2K} = D_3 z_0 + D_4, \qquad (A95)$$

Continuing from (A91) by the usual procedure we get the following final solution:

$$x_0 = C_2 z_0, \qquad (A96)$$

$$\mathcal{F}_1 = \frac{D_3 x + C_2 D_4}{D_4 + D_3 z}, \qquad (A97)$$

$$\mathcal{F}_3 = \frac{x - C_2 z}{D_4 + D_3 z}, \qquad (A98)$$

and we next have to verify (A8). We first try

Case 1.2.1.1: $\mathcal{G}_2 \neq 0$

Then (A8) can be written as

$$\frac{y - y_0}{\mathcal{G}_2} = (z - z_0)\frac{\mathcal{G}_1}{\mathcal{G}_2} + \frac{1}{2KR} \qquad (A99)$$

$$+\frac{1}{2}\left[(x - C_2 z_0)^2 + (y - y_0)^2 + (z - z_0)^2\right] + \frac{1}{2K}\frac{\mathcal{G}_3}{\mathcal{G}_2},$$

Using (A95) and (A96) we take the second derivative of this by y and t. Since we are in Case 1.2.1, where $z_{0,t} \neq 0$, we get

$$\left(\frac{\mathcal{G}_1}{\mathcal{G}_2}\right)_{,y} - D_3\left(\frac{\mathcal{G}_3}{\mathcal{G}_2}\right)_{,y} = \frac{y_{0,t}}{z_{0,t}}\left[\left(\frac{1}{\mathcal{G}_2}\right)_{,y} - 1\right]. \quad (A100)$$

Case 1.2.1.1.1: $(y_{0,t}/z_{0,t})_{,t} \neq 0$

Then (A100) solves as

$$\frac{1}{\mathcal{G}_2} = y + \mathcal{G}_4(x,z), \qquad (A101)$$

$$\mathcal{G}_1 = D_3\mathcal{G}_3 + \frac{\mathcal{G}_5(x,z)}{y + \mathcal{G}_4(x,z)}. \qquad (A102)$$

Proceeding from (A99) by the usual routine we find $\mathcal{G}_3$ and the arbitrary functions of (x,z), and we end up with

$$\frac{1}{R}(D_4 + D_3 z_0) + \frac{1}{2}\left[y_0{}^2 + \left(1 + C_2{}^2\right)z_0{}^2\right]$$
$$+ C_4 y_0 + C_5 z_0 = E_1. \qquad (A103)$$

This is equivalent to the subcase $D_1 = C_1 = 0$ of (A39) under the coordinate transformation $(x,y) = (y',x')$ that interchanges x_0 with y_0. The remark under (A75) about the transformation of $\mathcal{F}_1$ and $\mathcal{G}_1$ applies also here.

We go back to (A100) and consider

Case 1.2.1.1.2: $(y_{0,t}/z_{0,t})_{,t} = 0$

Then

$$y_0 = C_3 z_0, \qquad (A104)$$

$$\frac{\mathcal{G}_1}{\mathcal{G}_2} = D_3\frac{\mathcal{G}_3}{\mathcal{G}_2} + C_3\left(\frac{1}{\mathcal{G}_2} - y\right) + \mathcal{G}_4(x,z). \quad (A105)$$

In the usual way this leads from (A99) to

$$\frac{1}{R}(D_4 + D_3 z_0) + \frac{1}{2}\left(1 + C_2{}^2 + C_3{}^2\right)z_0{}^2 - C_5 z_0 = E_1, \qquad (A106)$$

which is formally the subcase $C_1 = C_4 = D_1 = 0, x_0 = C_2 z_0$ of (A39), but belongs to the class with $G_x = G_y = G_z = 0$ considered under Case 2.

This exhausts Case 1.2.1.1., so we consider

Case 1.2.1.2: $\mathcal{G}_2 = 0$

Then (A8) becomes, using (A95)

$$y - y_0 = (z - z_0)\,\mathcal{G}_1 + (D_4 + D_3 z_0)\,\mathcal{G}_3. \qquad (A107)$$

By the normal routine this leads to

$$y_0 = C_3 z_0, \qquad (A108)$$

$$\mathcal{G}_1 = \frac{D_3 y + C_3 D_4}{D_4 + D_3 z}, \qquad (A109)$$

$$\mathcal{G}_3 = \frac{y - C_3 z}{D_4 + D_3 z}, \qquad (A110)$$

which does not impose any limitation on (A95) – (A98) and leaves R undetermined. Thus, this subcase has a nontrivial FLRW limit. We have

$$V = \frac{1}{R} + \frac{k}{4R} \left[x^2 + y^2 + z^2 - 2\left(C_2 x + C_3 y + z\right) z_0 \right.$$
$$\left. + \left(1 + C_2{}^2 + C_3{}^2\right) z_0{}^2 \right], \qquad (A111)$$

$$k = \frac{2R}{D_4 + D_3 z_0}. \qquad (A112)$$

The flat FLRW model is contained here in the limit $(D_4, D_2) \to \infty$. In the general case, an orthogonal transformation of (x, y, z) may be used to achieve $C_2 = C_3 = 0$ (z_0 is then transformed to $\widetilde{z}_0 = \sqrt{1 + C_2{}^2 + C_3{}^2} z_0$), and then the model is seen to be axially symmetric, with the orbits of symmetry in the new (x, y) plane. Assuming $C_2 = C_3 = 0$ and using (A97), (A109) and (A111) in (A1) we get

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{D_3 x}{D_4 + D_3 z}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{D_3 y}{D_4 + D_3 z} \qquad (A113)$$

as the equations defining the RLPs in this case. As is easy to see, they obey $y\mathrm{d}x/\mathrm{d}z - x\mathrm{d}y/\mathrm{d}z = 0$, and so are, in the (x, y) surface, straight lines passing through the symmetry axis $x = y = 0$.

Thereby, Case 1.2.1 is exhausted, so we go back to (A92) to consider

Case 1.2.2: $z_{0,t} = 0$

This is equivalent to $z_0 = 0$. Then (A92) leads to two further subcases:

Case 1.2.2.1: $(1/2K)_{,t} \neq 0$

By the usual method we obtain from (A92) and (A91):

$$\mathcal{F}_1 = \frac{x - E_4}{z}, \qquad \mathcal{F}_3 = E_3, \qquad \frac{E_3}{2K} = E_4 - x_0, \quad (A114)$$

with R undetermined. This is the final solution of (A7).

For checking (A8) we have to consider separately

Case 1.2.2.1.1: $E_3 \neq 0$

Then (A114) determines K, and from the definition of Case 1.2.2.1 it follows that $x_{0,t} \neq 0$. But to continue, we have to separately consider $\mathcal{G}_2$ being zero or not.

Case 1.2.2.1.1.1: $\mathcal{G}_2 \neq 0$

Then (A8) is written as

$$\frac{y - y_0}{\mathcal{G}_2} = z\frac{\mathcal{G}_1}{\mathcal{G}_2} + \frac{1}{2KR} \qquad (A115)$$
$$+ \frac{1}{2}\left[(x - x_0)^2 + (y - y_0)^2 + z^2\right] + \frac{1}{E_3}(E_4 - x_0)\frac{\mathcal{G}_3}{\mathcal{G}_2}.$$

Taking the derivative of this by y and t, and knowing that $x_{0,t} \neq 0$, we get

$$\frac{y_{0,t}}{x_{0,t}}\left[\left(\frac{1}{\mathcal{G}_2}\right)_{,y} - 1\right] = \frac{1}{E_3}\left(\frac{\mathcal{G}_3}{\mathcal{G}_2}\right)_{,y}. \qquad (A116)$$

Case 1.2.2.1.1.1.1: $(y_{0,t}/x_{0,t})_{,t} \neq 0$

Then

$$\frac{1}{\mathcal{G}_2} = y + \mathcal{G}_4(x, z), \qquad \frac{\mathcal{G}_3}{\mathcal{G}_2} = \mathcal{G}_5(x, z). \qquad (A117)$$

Proceeding from (A115) in the usual way we obtain

$$\frac{1}{E_3 R}(E_4 - x_0) + \frac{1}{2}\left(x_0{}^2 + y_0{}^2\right) + C_4 x_0 - C_5 y_0 = E_1. \qquad (A118)$$

This is the subcase of (A39) that results when we take $C_1 = C_3 = D_3 = 0$ in (A39) and interchange y_0 with z_0. How the new $\mathcal{F}_1$ and $\mathcal{G}_1$ are calculated after such a transformation is explained under (A61). To have a consistent naming of the constants, one must take in (A39) $(D_1, D_4) = (-1/E_3, E_4/E_3)$.

We go back to (A116) and consider

Case 1.2.2.1.1.1.2: $(y_{0,t}/x_{0,t})_{,t} = 0$

Then from the definition above

$$y_0 = C_2 x_0, \qquad (A119)$$

and by integrating (A116)

$$\frac{\mathcal{G}_3}{\mathcal{G}_2} = C_2 E_3\left(\frac{1}{\mathcal{G}_2} - y\right) + E_3\mathcal{G}_4(x, z). \qquad (A120)$$

By continuing from (A115) in the usual way we get finally

$$\frac{1}{E_3 R}(E_4 - x_0) + \frac{1}{2}\left(1 + C_2{}^2\right) x_0{}^2 + C_4 x_0 = E_1. \quad (A121)$$

This is formally the subcase $y_0 = C_2 x_0$ of (A118), again with some renaming of the constants. However, it has $G_x = G_y = G_z = 0$, so in fact it belongs in Case 2 considered further on.

This completes Case 1.2.2.1.1.1, so we go back to (A114) to consider

Case 1.2.2.1.1.2: $\mathcal{G}_2 = 0$

Then (A8) is written as

$$y - y_0 = z\mathcal{G}_1 + \frac{1}{E_3}(E_4 - x_0)\mathcal{G}_3. \qquad (A122)$$

Case 1.2.2.1.1.2.1: $x_{0,t} \neq 0$

Then the final solution of (A122) is

$$y_0 = C_2 x_0, \qquad \mathcal{G}_1 = \frac{1}{z}(y - E_4 C_2), \qquad \mathcal{G}_3 = C_2 E_3, \qquad (A123)$$

where C_2 is allowed to be zero. This imposes no additional conditions on (A114), and R remains undetermined. The function V is in this case

$$V = \frac{1}{R} + \frac{E_3}{2\left(E_4 - x_0\right)}\left[x^2 + y^2 + z^2\right.$$
$$\left. -2x_0\left(x + C_2 y\right) + \left(1 + C_2{}^2\right)x_0{}^2\right]. \quad \text{(A124)}$$

This is equivalent to (A111) – (A112), which is seen when we transform C_2 to 0 by a rotation in the (x,y) plane. Subsequently, the new (x,z) have to be transformed by $(x,z) = (z',x')$. How the new $\mathcal{F}_1$ and $\mathcal{G}_1$ are calculated after such a transformation is explained under (A84).

We go back to (A122) to consider

Case 1.2.2.1.1.2.2: $x_{0,t} = 0$

Then (A122) implies $y_{0,t} = 0$. With $x_{0,t} = y_{0,t} = z_{0,t} = 0$ we are then in the spherically symmetric subcase of the Stephani solution, but not in its full generality. Since (A114) still applies, with $x_0 = $ constant we have $K \equiv k/(4R) = $ constant, and this corresponds to the subcase $A_2 = 0$ of (5.10). Since this has already been investigated, we go back to (A114) to consider

Case 1.2.2.1.2: $E_3 = 0$

Then (A114) leaves K still arbitrary, but implies $x_0 = E_4$. By a transformation of x this can be changed to

$$x_0 = E_4 = 0. \quad \text{(A125)}$$

With this, we go on to consider (A8). Recall that we are still in Case 1.2.2, where also $z_0 = 0$.

Case 1.2.2.1.2.1: $\mathcal{G}_2 \neq 0$

Then (A8) is written as

$$\frac{y - y_0}{\mathcal{G}_2} = z\frac{\mathcal{G}_1}{\mathcal{G}_2} + \frac{1}{2KR} \quad \text{(A126)}$$
$$+ \frac{1}{2}\left[x^2 + \left(y - y_0\right)^2 + z^2\right] + \frac{1}{2K}\frac{\mathcal{G}_3}{\mathcal{G}_2}.$$

Taking the derivative of this by y and t we get

$$-y_{0,t}\left(\frac{1}{\mathcal{G}_2}\right)_{,y} = -y_{0,t} + \left(\frac{1}{2K}\right)_{,t}\left(\frac{\mathcal{G}_3}{\mathcal{G}_2}\right)_{,y}. \quad \text{(A127)}$$

We are still in Case 1.2.2.1, where $(1/2K)_{,t} \neq 0$, so we divide (A127) by $(1/2K)_{,t}$ and consider first

Case 1.2.2.1.2.1.1: $\left[y_{0,t}/\left(1/2K\right)_{,t}\right]_{,t} \neq 0$

Then (A127) implies $(1/\mathcal{G}_2)_{,y} - 1 = (\mathcal{G}_3/\mathcal{G}_2)_{,y} = 0$. This leads to a simple coordinate transform of (A84) (z_0 replaced by y_0), which is a subcase of (A64).

We go back to (A127) and consider

Case 1.2.2.1.2.1.2: $\left[y_{0,t}/\left(1/2K\right)_{,t}\right]_{,t} = 0$

We have then $y_0 = D_1/(2K) + D_2$. We do not have to consider the case $D_1 = 0$ because this would mean $y_0 = 0$ in addition to $x_0 = z_0 = 0$, and we would be back in the spherically symmetric Stephani solution, considered in Sec. V. So the new situation arises only when $D_1 \neq 0$, and then we can rewrite the last formula as

$$\frac{1}{2K} = D_1 + D_2 y_0, \quad \text{(A128)}$$

obtaining from (A127)

$$\frac{1}{\mathcal{G}_2} = y - D_2\frac{\mathcal{G}_3}{\mathcal{G}_2} + \mathcal{G}_4(x,z). \quad \text{(A129)}$$

This leads to the subcase $y_0 = 0$ of (A118), with x_0 subsequently changed to y_0. This again has $G_x = G_y = G_z = 0$ and belongs in Case 2.

Case 1.2.2.1.2.1 is now exhausted, so we go back to (A125) and consider

Case 1.2.2.1.2.2: $\mathcal{G}_2 = 0$

Then (A8) becomes

$$y - y_0 = z\mathcal{G}_1 + \frac{1}{2K}\mathcal{G}_3. \quad \text{(A130)}$$

This leads to

$$\mathcal{G}_1 = \frac{y - C_4}{z}, \qquad \mathcal{G}_3 = C_5, \qquad \frac{C_5}{2K} = C_4 - y_0. \quad \text{(A131)}$$

When $C_5 \neq 0$ this is equivalent to (A123) – (A124), with the constants in (A131) related to those in (A123) by $(C_4, C_5) = C_2(E_4, E_3)$. When $C_5 = 0$, we are back in the spherically symmetric subclass.

This completes Case 1.2.2.1. We have to go back as far as (A92) and consider

Case 1.2.2.2: $(1/2K)_{,t} = 0$

With $z_{0,t} = 0$ now being considered (Case 1.2.2), (A92) is fulfilled identically, and (A91) immediately implies $x_{0,t} = 0$. Taking $x_0 = z_0 = 0$ and $1/2K = B = $ constant in (A91) we get

$$x = z\mathcal{F}_1 + B\mathcal{F}_3, \quad \text{(A132)}$$

with no limitation on R, and this is the final solution of (A7). Running this through (A8) with $\mathcal{G}_2 \neq 0$ we obtain

$$\frac{B}{R} + \frac{1}{2}y_0{}^2 + C_4 y_0 = C_5. \quad \text{(A133)}$$

This case again has $G_x = G_y = G_z = 0$, and so belongs in Case 2.

Running (A132) through (A8) with $\mathcal{G}_2 = 0$ we immediately get $y_{0,t} = 0$ and no limitation on R. This is a subcase of the spherically symmetric Stephani solution, the same one that we obtained in Case 1.2.2.1.1.2.2.

This completes Case 1.2, and the whole Case 1. So we go back to (11.5) – (11.6) and consider

Case 2: $G_z = 0$

Then we immediately have $G_x = G_y = 0$ in (11.5) – (11.6). This means that (11.5) and (11.6) are fulfilled

for any dx/dz and dy/dz, i.e. that all null geodesics are RLPs in this case.

Multiplying the equations $G_x = G_y = G_z = 0$ by $V_{,t}^2$ we turn them into polynomials in (x, y, z). Taking the coefficients of x^2 in each polynomial we get[7]

$$-2KK_{,tt}\,x_{0,t} + 2Kx_{0,tt}K_{,t} + 4K_{,t}^2 x_{0,t} = 0, \quad \text{(A134)}$$

$$2KK_{,tt}\,y_{0,t} - 2Ky_{0,tt}K_{,t} - 4K_{,t}^2 y_{0,t} = 0, \quad \text{(A135)}$$

$$2KK_{,tt}\,z_{0,t} - 2Kz_{0,tt}K_{,t} - 4K_{,t}^2 z_{0,t} = 0. \quad \text{(A136)}$$

These are easily integrated, but a few cases have to be considered separately. One solution is $x_{0,t} = y_{0,t} = z_{0,t} = 0$, but this is the spherically symmetric Stephani model that we investigated in Sec. V. So we assume that at least one of the functions (x_0, y_0, z_0) is non-constant. By a coordinate transformation we may choose this to be x_0. When $x_{0,t} \neq 0$, (A134) is integrated with the result

$$\frac{1}{2K} = D_1 x_0 + D_2, \quad \text{(A137)}$$

and then we consider

Case 2.1: $D_1 \neq 0$

In this case, (A135) – (A136) imply

$$y_0 = C_2 x_0, \qquad z_0 = C_3 x_0, \quad \text{(A138)}$$

with zero values of C_2 and C_3 allowed.

The terms free of (x, y, z) in the equations $G_x = Gy = G_z = 0$ are all the same:

$$\left(\frac{1}{R}\right)_{,t} K_{,tt} - \left(\frac{1}{R}\right)_{,tt} K_{,t}$$
$$-2KK_{,t}\left(x_{0,t}^2 + y_{0,t}^2 + z_{0,t}^2\right) = 0. \quad \text{(A139)}$$

Likewise, the terms linear in (x, y, z) are the same in all 3 equations, namely:

$$K\left(\frac{1}{R}\right)_{,tt} x_{0,t} - K\left(\frac{1}{R}\right)_{,t} x_{0,tt} - 2K_{,t}\left(\frac{1}{R}\right)_{,t} x_{0,t}$$

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[7] These formulae were calculated using the algebraic computer program Ortocartan [12, 13].

$$+2K^2 x_{0,t}\left(x_{0,t}^2 + y_{0,t}^2 + z_{0,t}^2\right) = 0, \quad \text{(A140)}$$

and the analogous equations with $x_{0,t}$ replaced by $y_{0,t}$ and $z_{0,t}$.

Using (A137) and (A138), eqs. (A139) and (A140) become identical and are integrated with the result:

$$\frac{1}{R}\left(D_1 x_0 + D_2\right) + \left(1 + C_2^2 + C_3^2\right) x_0^2 + C_4 x_0 = C_5. \quad \text{(A141)}$$

By an orthogonal coordinate transformation in the (x, y, z) space one can obtain $C_2 = C_3 = 0$. Then the model is seen to be axially symmetric, but is more general than the plane- or hyperbolically symmetric subcases.

Case 2.2: $D_1 = 0$

Then $K = 1/(2D_2) = $ constant, and (A134) – (A136) and (A139) are fulfilled identically. Instead of (A140) and its associated equations we then have their subcases $K_{,t} = 0$ which imply either $x_{0,t} = y_{0,t} = z_{0,t} = 0$ or (A138) plus the integral of (A140), which is

$$\frac{1}{R} + \frac{1}{2D_2}\left(1 + C_2^2 + C_3^2\right) x_0^2 + E_1 x_0 = E_2. \quad \text{(A142)}$$

This is the subcase $D_1 = 0$ of (A141).

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